A Laplace Type Problem for Irregular Lattice with Fundamental Cell Composed by Three Triangles and a Trapezium

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Abstract

In the previous papers, [1], [2], [3], [4],[5], [6], [7], [8], [9], [10], [11], [12], [13] and [14] the authors studies some Laplace problem for different lattices. In this paper we determine the probability that a random segment of constant length intersects a side of the lattice with fundamental cell represented in fig. 1.

Let \( \mathcal{R}(a) \) the lattice with fundamental cell \( C_0 \) represented in fig. 1.
With the rotations of this figure we can write that

\[ C_0 = C_{01} \cup C_{02} \cup C_{03} \cup C_{04} \]

and

\[
\begin{align*}
\widehat{ABE} &= \widehat{AEB} = \widehat{DEF} = \frac{\pi}{5}, & \widehat{CBE} = \widehat{DEB} = \widehat{EDF} = \widehat{EFD} &= \frac{2\pi}{5}, \\
\widehat{BCD} &= \widehat{CDE} = \frac{3\pi}{5}, & \widehat{BAG} &= \widehat{EAG} &= \frac{3\pi}{4}, \\
|BG| &= |EG| = a \cos \frac{\pi}{5}, & |BE| &= 2a \cos \frac{\pi}{5}, & |AG| &= a \sin \frac{\pi}{5}, \\
|DF| &= \frac{a}{2 \cos \frac{\pi}{5}}, & \text{area}C_0 &= \frac{a^2}{2} \sin \frac{\pi}{5} \left(2 \cos \frac{\pi}{5} + 1\right)^2.
\end{align*}
\]  

We want to compute the probability that a segment \( s \) of random position and of constant length \( l \), with \( l < \frac{a}{4 \cos \frac{\pi}{5}} \), intersects a side of the lattice \( \Re (a) \), i.e. the probability \( P_{int} \) that \( s \) intersects a side of the fundamental cell \( C_0 \). The position of the segment \( s \) is determined by center and by the angle \( \varphi \) that it formed with the line \( CF \).

In order to compute the probability \( P_{int} \) we consider the limit positions of the segment \( s \), for a fixed value of \( \varphi \), in the cells \( C_{0i} \), \( (i = 1, 2, 3, 4) \). We have the fig. 2
and the relations

\[ area \hat{C}_{01}(\varphi) = areaC_{01} - \sum_{j=1}^{5} areaa_j(\varphi), \] (4)

\[ area \hat{C}_{02}(\varphi) = areaC_{02} - \sum_{j=1}^{5} areab_j(\varphi), \] (5)

\[ area \hat{C}_{03}(\varphi) = areaC_{03} - \sum_{j=1}^{5} areac_j(\varphi), \] (6)

\[ area \hat{C}_{04}(\varphi) = areaC_{04} - \sum_{j=1}^{5} aread_j(\varphi). \] (7)

By fig.1 and fig.2 we have that:

\[ areaa_4(\varphi) = \frac{l^2}{4} \sin 2\varphi, \]

\[ areaa_3(\varphi) = \frac{al}{2} \cos \frac{\pi}{5} \sin \varphi - \frac{l^2}{4} \cos \varphi, \]

\[ areaa_1(\varphi) = \frac{l^2 \cos \varphi \sin \left(\frac{\pi}{5} - \varphi\right)}{2 \cos \frac{\pi}{5}}, \]
\[ areaa_2 (\varphi) = \frac{a l}{2} \sin \left( \frac{\pi}{5} - \varphi \right) - \frac{l^2 \cos \varphi \sin \left( \frac{\pi}{5} - \varphi \right)}{2 \cos \frac{\pi}{5}}, \]

\[ areaa_5 (\varphi) = \frac{a l}{2} \sin \frac{\pi}{5} \cos \varphi - \frac{l^2}{4} \sin 2\varphi - \frac{l^2 \cos \varphi \sin \left( \frac{\pi}{5} - \varphi \right)}{2 \cos \frac{\pi}{5}}. \]

We obtain that:

\[ area\hat{C}_01 = areaC_{01} - A_1 (\varphi), \]

where

\[ A_1 (\varphi) = a l \sin \frac{\pi}{5} \cos \varphi - \frac{l^2}{4} \tan \frac{\pi}{5} (1 + \cos 2\varphi). \]

In order to compute \( area\hat{C}_2 (\varphi) \) we have that:

\[ areab_4 (\varphi) = \frac{l^2 \sin \varphi \sin \left( \frac{\pi}{5} + \varphi \right)}{2 \sin \frac{\pi}{5}}, \]

\[ areab_3 (\varphi) = \frac{a l}{2} \cos \frac{\pi}{5} \sin \varphi - \frac{l^2 \sin \varphi \sin \left( \frac{\pi}{5} + \varphi \right)}{2 \sin \frac{\pi}{5}}, \]

\[ areab_1 (\varphi) = \frac{l^2 \cos \varphi \sin \left( \frac{\pi}{5} + \varphi \right)}{2 \cos \frac{\pi}{5}}, \]

\[ areab_2 (\varphi) = \frac{a l}{2} \sin \frac{\pi}{5} \cos \varphi - \frac{l^2 \cos \varphi \sin \left( \frac{\pi}{5} + \varphi \right)}{\cos \frac{\pi}{5}}, \]

\[ areab_5 (\varphi) = \frac{a l}{2} \sin \left( \frac{\pi}{5} + \varphi \right) - \]

\[ \frac{l^2}{2 \sin \frac{2\pi}{5}} \left( 1 - \cos \frac{2\pi}{5} \cos 2\varphi + \frac{1}{2} \sin \frac{2\pi}{5} \sin 2\varphi \right). \]

We obtain that:

\[ area\hat{C}_02 = areaC_{02} - A_2 (\varphi), \]

where

\[ A_2 (\varphi) = a l \sin \left( \frac{\pi}{5} + \varphi \right) - \frac{l^2}{2 \sin \frac{2\pi}{5}} \left( 1 - \cos \frac{2\pi}{5} \cos 2\varphi + \frac{1}{2} \sin \frac{2\pi}{5} \sin 2\varphi \right). \]

To compute \( area\hat{C}_03 (\varphi) \) we have that

\[ areac_4 (\varphi) = \frac{l^2 \sin \varphi \sin \left( \frac{2\pi}{5} - \varphi \right)}{2 \sin \frac{2\pi}{5}}. \]
A Laplace type problem for irregular lattice

\[ \text{area}_{c_3}(\varphi) = \frac{a}{2} \sin \varphi - \frac{l^2 \sin \varphi \sin \left( \frac{2\pi}{5} - \varphi \right)}{2 \sin \frac{2\pi}{5}}, \]

\[ \text{area}_{c_1}(\varphi) = \frac{l^2 \sin \varphi \sin \left( \frac{2\pi}{5} + \varphi \right)}{2 \sin \frac{2\pi}{5}}, \]

\[ \text{area}_{c_2}(\varphi) = \frac{a}{2} \sin \left( \frac{2\pi}{5} + \varphi \right) - \frac{l^2 \sin \varphi \sin \left( \frac{2\pi}{5} + \varphi \right)}{2 \sin \frac{2\pi}{5}}, \]

\[ \text{area}_{c_5}(\varphi) = \frac{a}{2} \sin \left( \frac{2\pi}{5} - \varphi \right) - \frac{l^2 \sin \varphi \sin \left( \frac{2\pi}{5} - \varphi \right)}{2 \sin \frac{2\pi}{5}}, \]

\[ \text{area}_{c_6}(\varphi) = al \cos \frac{\pi}{5} \sin \varphi - \frac{l^2 \sin \varphi \sin \left( \frac{2\pi}{5} + \varphi \right)}{2 \sin \frac{2\pi}{5}}. \]

We obtain that

\[ \text{area}_{\hat{c}_03}(\varphi) - A_3(\varphi), \]

where

\[ A_3(\varphi) = \frac{a}{2} \left[ 2 \sin \frac{2\pi}{5} \cos \varphi + \left( 1 + 2 \cos \frac{\pi}{5} \right) \sin \varphi \right] - \frac{l^2}{2} \sin 2\varphi. \]

To compute now the \( \text{area}_{\hat{c}_04}(\varphi). \)

\[ \text{area}_{d_4}(\varphi) = \frac{l^2 \sin \varphi \sin \left( \frac{2\pi}{5} + \varphi \right)}{2 \sin \frac{2\pi}{5}}, \]

\[ \text{area}_{d_3}(\varphi) = \frac{a \sin(\varphi)}{4 \cos \frac{\pi}{5}} - \frac{l \sin \left( \frac{2\pi}{5} + \varphi \right)}{\sin \frac{2\pi}{5}}, \]

\[ \text{area}_{d_1}(\varphi) = \frac{l^2 \sin \left( \frac{2\pi}{5} - \varphi \right) \sin \left( \frac{2\pi}{5} + \varphi \right)}{2 \sin \frac{\pi}{5}}, \]

\[ \text{area}_{d_2}(\varphi) = \frac{a}{2} \sin \left( \frac{2\pi}{5} - \varphi \right) - \frac{l^2 \sin \left( \frac{2\pi}{5} - \varphi \right) \sin \left( \frac{2\pi}{5} + \varphi \right)}{2 \sin \frac{\pi}{5}}, \]

\[ \text{area}_{d_5}(\varphi) = \frac{a}{2} \sin \left( \frac{2\pi}{5} + \varphi \right) - \frac{l^2 \sin \left( \frac{2\pi}{5} - \varphi \right) \sin \left( \frac{2\pi}{5} + \varphi \right)}{2 \sin \frac{2\pi}{5}} - \frac{l^2 \sin \varphi \sin \left( \frac{2\pi}{5} + \varphi \right)}{2 \sin \frac{2\pi}{5}}. \]
We obtain that:

\[
\text{area } \hat{C}_{04} (\varphi) = \text{area } C_{04} - \mathcal{A}_4 (\varphi),
\]

where

\[
\mathcal{A}_4 (\varphi) = a l \left( \sin \frac{2\pi}{5} \cos \varphi + \sin \varphi \frac{2\pi}{4 \cos \frac{2\pi}{5}} \right) - \frac{l^2}{4 \sin \frac{2\pi}{5}} \left[ \left( 2 \cos \frac{\pi}{5} - \cos \frac{2\pi}{5} \right) \cos 2\varphi + \sin \frac{2\pi}{5} \sin 2\varphi + 4 \cos^2 \frac{\pi}{5} + 1 \right].
\]

Denoting with \( M_i \), \( i = 1, 2, 3, 4 \), the set of segments \( s \) that have the center in the cell \( C_{0i} \) and with \( N_i \) the set of the segments \( s \) completely in \( C_{0i} \), we have that [16]:

\[
P_{\text{int}} = 1 - \frac{\sum_{i=1}^{4} \mu (N_i)}{\sum_{i=1}^{4} \mu (M_i)}, \tag{8}
\]

where \( \mu \) is the Lebesgue measure in the euclidean plane.

In order to compute \( \mu (M_i) \) and \( \mu (N_i) \) we use the kinematic measure of Poincaré [15]:

\[
dK = dx \wedge dy \wedge d\varphi,
\]

where \( x, y \) are the coordinates of middle point of \( s \) and \( \varphi \) the fixed angle.

We can write:

\[
\mu (M_i) = \int_0^{\frac{\pi}{5}} d\varphi \int\int_{\{ (x,y) \in \hat{C}_{0i} \}} dx dy = \int_0^{\frac{\pi}{5}} (\text{area } \hat{C}_{0i}) d\varphi = \frac{\pi}{5} \text{area } \hat{C}_{0i}, \quad (i = 1, 2, 3, 4)
\]

and

\[
\mu (N_i) = \int_0^{\frac{\pi}{5}} d\varphi \int\int_{\{ (x,y) \in \hat{C}_{0i} \}} dx dy = \int_0^{\frac{\pi}{5}} \left[ \text{area } \hat{C}_{0i} (\varphi) \right] d\varphi =
\]

\[
\int_0^{\frac{\pi}{5}} \left[ \text{area } C_{0i} - \mathcal{A}_i (\varphi) \right] d\varphi = \frac{\pi}{5} \text{area } C_{0i} - \int_0^{\frac{\pi}{5}} \mathcal{A}_i (\varphi) d\varphi, \quad (i = 1, 2, 3, 4).
\]

By these two relations give us:

\[
\sum_{i=1}^{4} \mu (M_i) = \frac{\pi}{5} \text{area } C_0, \tag{9}
\]
and
\[
\sum_{i=1}^{4} \mu(N_i) = \frac{\pi}{5} \text{area} C_0 - \int_{0}^{\pi} \left[ \sum_{i=1}^{4} A_i(\varphi) \right] d\varphi. \tag{10}
\]
follow that:
\[
\sum_{i=1}^{4} A_i(\varphi) = al \left[ \sin \frac{\pi}{5} \left( 2 + 3 \cos \frac{\pi}{5} \right) \cos \varphi + \frac{8 \cos^{2} \frac{\pi}{5} + 2 \cos \frac{\pi}{5} + 1}{4 \cos \frac{\pi}{5}} \sin \varphi \right] -
\]
\[
\frac{l^{2}}{4} \left[ \left( \tan \frac{\pi}{5} - 3 \csc \frac{2\pi}{5} + \frac{1}{\sin \frac{\pi}{5}} \right) \cos 2\varphi + 2 \sin 2\varphi + 4 \cos^{2} \frac{\pi}{5} + \tan \frac{\pi}{5} + \frac{3}{\sin \frac{2\pi}{5}} \right],
\]
then
\[
\int_{0}^{\pi} \left[ \sum_{i=1}^{4} A_i(\varphi) \right] d\varphi = al \left( \frac{9}{4} - 3 \cos^{3} \frac{\pi}{5} - 4 \cos^{2} \frac{\pi}{5} - \frac{9}{2} \cos \frac{\pi}{5} + \frac{1}{4 \cos \frac{\pi}{5}} \right) -
\]
\[
\frac{l^{2}}{4} \left[ \left( \frac{9}{2} + \cos \frac{\pi}{5} - 6 \cos^{2} \frac{\pi}{5} + \pi \frac{4 \cos^{2} \frac{\pi}{5} + \tan \frac{\pi}{5} + \frac{3}{\sin \frac{2\pi}{5}} \right) \right]. \tag{11}
\]

We obtain that:
\[
P_{int} = \frac{10}{\pi a^{2} \sin \frac{\pi}{5} \left( 2 \cos \frac{\pi}{5} + 1 \right)^{2}} \left\{ al \left( \frac{9}{4} - 3 \cos^{3} \frac{\pi}{5} - 4 \cos^{2} \frac{\pi}{5} - \frac{9}{2} \cos \frac{\pi}{5} + \right.ight.
\]
\[
\frac{1}{4 \cos \frac{\pi}{5}} \left. - \frac{l^{2}}{4} \left[ \left( \frac{9}{2} + \cos \frac{\pi}{5} - 6 \cos^{2} \frac{\pi}{5} + \pi \frac{4 \cos^{2} \frac{\pi}{5} + \tan \frac{\pi}{5} + \frac{3}{\sin \frac{2\pi}{5}} \right) \right] \right\}.
\]

References


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