On Pseudo $BCH$-algebras

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Abstract

As a generalization of $BCH$-algebras, the notion of pseudo $BH$-algebra is introduced, and some of their properties are investigated. The notions of pseudo subalgebra, pseudo ideals, and pseudo atoms in pseudo $BCH$-algebras are introduced. Characterizations of their properties are provided.

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Keywords: pseudo atom, pseudo subalgebra, pseudo ideal, pseudo $BCH$-algebra

1 Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: $BCK$-algebras and $BCI$-algebras([6,7]). It is known that the class of $BCK$-algebras is a proper subclass of the class of $BCI$-algebras. Q. P. Hu and X. Li([4,5]) introduced a wide class of abstract algebras: $BCH$-algebras. They have shown that the class of $BCI$-algebras is a proper subclass of the class of

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BCH-algebras. BCK-algebras have several connections with other areas of investigation, such as: lattice ordered groups, MV-algebras, Wajsberg algebras, and implicative commutative semigroups. J. M. Font et al.([2]) have discussed Wajsberg algebras which are term-equivalent to MV-algebras. D. Mundici([12]) proved MV-algebras are categorically equivalent to bounded commutative BCK-algebra, and J. Meng([10]) proved that implicative commutative semigroups are equivalent to a class of BCK-algebras. G. Georgescu and A. Iorgulescu([3]) introduced the notion of a pseudo BCK-algebra. Y. B. Jun characterized pseudo BCK-algebras. He found conditions for a pseudo BCK-algebra to be ∧-semi-lattice ordered. Y. B. Jun, H.S. Kim, J. Neggers([8]) introduced the notion of a pseudo d-algebra as a generalization of the idea of a d-algebra.

In this paper, we introduce the notion of pseudo BCH-algebra as a generalization of BCH-algebra and investigate some of their properties. We also define the notions of pseudo subalgebra, pseudo ideals, and pseudo atoms in pseudo BCH-algebras and provide characterizations of their properties in pseudo BCH-algebras.

2 Preliminaries

By a BCH-algebra([3]), we mean an algebra \((X; \star, 0)\) of type (2,0) satisfying the following conditions:

(I) \(x \star x = 0\),

(II) \((x \star y) \star z = (x \star z) \star y\),

(III) \(x \star y = 0\) and \(y \star x = 0\) imply \(x = y\), for all \(x, y \in X\).

For brevity, we also call \(X\) a BCH-algebra. In \(X\) we can define a binary operation “\(\leq\)” by \(x \leq y\) if and only if \(x \star y = 0\). The following hold([3]):

(1) \((x \star (x \star y)) \star y = 0\),

(2) \(x \star 0 = 0\) implies \(x = 0\),

(3) \(x \star 0 = x\), for all \(x, y \in X\).

A non-empty subset \(S\) of a BCH-algebra \(X\) is called a subalgebra of \(X\) if, for any \(x, y \in S\), \(x \star y \in S\), i.e., \(S\) is a closed under binary operation.

Definition 2.1. A non-empty subset \(A\) of a BCH-algebra \(X\) is called an ideal([3]) of \(X\) if it satisfies:

(I1) \(0 \in I\).
(I2) $y, x \ast y \in A$ imply $x \in I$ for all $y \in X$.

A non-empty subset $A$ of a $BCH$-algebra $X$ is called a closed ideal ([1]) of $X$ if it satisfies: (I2) and

(I3) $0 \ast x \in A$ for all $x \in A$.

Since $A$ is non-empty so there is an element $x \in A$. Further (I3) gives $0 \ast x \in A$, whereas (I2) gives $0 \in A$. Moreover if $x, y \in A$, then $(x \ast y) \ast x = 0 \ast y \in A$ and (I2) gives $x \ast y \in A$. Thus every closed ideal in a $BCH$-algebra $X$ is a subalgebra but converse is not true (see [1]).

3 Pseudo $BCH$-algebras

Definition 3.1. A pseudo $BCH$-algebra is a non-empty set $X$ with a constant $0$ and two binary operations “$\ast$” and “$\odot$” satisfying the following axioms: for any $x, y, z \in X$,

(P1) $x \ast x = x \odot x = 0$;

(P2) $(x \ast y) \odot z = (x \odot z) \ast y$;

(P3) $x \ast y = y \odot x = 0$ imply $x = y$.

For brevity, we also call $X$ a pseudo $BCH$-algebra. In $X$ we can define a binary operation “$\preceq$” by $x \preceq y$ if and only if $x \ast y = 0$ if and only if $x \odot y = 0$. Note that if $(X; \ast, 0)$ is a $BCH$-algebra, then letting $x \odot y := x \ast y$, produces a pseudo $BCH$-algebra $(X; \ast, \odot, 0)$. Hence every $BCH$-algebra is a pseudo $BCH$-algebra in a natural way.

Definition 3.2. Let $(X; \ast, \odot, 0)$ be a pseudo $BCH$-algebra and let $\emptyset \neq I \subseteq X$. $I$ is called a pseudo subalgebra of $X$ if $x \ast y, x \odot y \in I$ whenever $x, y \in I$. $I$ is called a pseudo ideal of $X$ if it satisfies

(Pi1) $0 \in I$;

(Pi2) $x \ast y, x \odot y \in I$ and $y \in I$ imply $x \in I$ for all $x, y \in X$.

Example 3.3. Let $X := \{0, a, b, c\}$ be a set with the following Cayley tables:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
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<tbody>
<tr>
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<td>0</td>
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<td>a</td>
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<td>0</td>
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<td>b</td>
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<td>b</td>
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<td>c</td>
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<td>c</td>
<td>0</td>
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<table>
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<td>0</td>
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<td>c</td>
<td>c</td>
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<td>0</td>
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</tr>
</tbody>
</table>
Then \((X; *, 0)\) and \((X; \odot, 0)\) are not BCH-algebras, since \((b*a) * c = a \neq 0 = (b*c)*a\) and \((b \odot a)c = 0 \neq c = (b \odot c) \odot a\). It is easy to check that \((X; *, \odot, 0)\) is a pseudo BCH-algebra. Let \(I := \{0, a\}\). Then \(I\) is both a pseudo subalgebra of \(X\) and a pseudo ideal of \(X\). Let \(J := \{0, a, c\}\). Then \(J\) is a pseudo subalgebra of \(X\), but it is not a pseudo ideal of \(X\) since \(b \odot c = c \in J\) and \(b \ast c = a \in J\), but \(b \notin J\).

**Proposition 3.4.** Let \(I\) be a pseudo ideal of a pseudo BCH-algebra \(X\). If \(x \in I\) and \(y \leq x\), then \(y \in I\).

**Proof.** Assume that \(x \in I\) and \(y \leq x\). Then \(y \ast x = 0\) and \(y \odot x = 0\). By (P1) and (P2), we have \(y \in I\). \(\square\)

**Proposition 3.5.** Let \((X; *, \odot, 0)\) be a pseudo BCH-algebra. Then the following hold: for all \(x, y, z \in X\).

(i) \(x \leq 0\) implies \(x = 0\),

(ii) \(x \ast (x \odot y) \leq y, x \odot (x \ast y) \leq y\),

(iii) \(x \ast 0 = x \odot 0 = x\),

(iv) \(x \ast y \leq z \iff x \odot z \leq y\),

(v) \(0 \ast (x \ast y) = (0 \odot x) \odot (0 \ast y)\),

(vi) \(0 \odot (x \ast y) = (0 \ast x) \odot (0 \odot y)\),

(vii) \(0 \ast x = 0 \odot x\).

**Proof.** (i) Let \(x \leq 0\). Then \(x \ast 0 = x \odot 0 = 0\). It follows from (P1) and (P2) that \(0 \odot x = (x \ast 0) \odot x = (x \odot x) \ast 0 = 0 \ast 0 = 0\) and \(0 \ast x = (x \odot 0) \ast x = (x \ast x) \odot 0 = 0 \odot 0 = 0\). Using (P3), we have \(x = 0\).

(ii) By (P2) and (P1), we obtain \([x \ast (x \odot y)] \odot y = (x \odot y) \ast (x \ast y) = 0\) and \([x \odot (x \ast y)] \ast y = (x \ast y) \odot (x \ast y) = 0\). Hence \(x \ast (x \odot y) \leq y\) and \(x \ast (x \ast y) \leq y\).

(iii) Using (P2) and (P1), we have \((x \ast 0) \odot x = (x \odot x) \ast 0 = 0 \ast 0 = 0\). By (ii), we obtain \(x \ast (x \odot 0) \leq 0\). It follows from (i) that \(x \ast (x \odot 0) = 0\). Hence \(x \ast 0 = x\) by (P3). By a similar way, we have \(x \odot 0 = x\).

(iv) \(x \ast y \leq z \iff (x \odot y) \odot z = 0 \iff (x \odot z) \ast y = 0 \iff x \odot z \leq y\).

(v) For any \(x, y \in X\), by (P1) and (P2) we have \((0 \odot x) \odot (0 \ast y) = [(x \ast y) \ast (x \ast y)] \odot (0 \odot y) = [(x \odot x) \odot (x \ast y) \ast (x \ast y)] \odot (0 \ast y) = [(0 \odot y) \odot (0 \ast y) \ast (x \ast y) \odot (0 \odot y) = [(0 \ast y) \odot (0 \ast y) \ast (x \ast y) \odot (0 \odot y) = [(x \odot x) \odot (x \ast y) \odot (x \ast y)] \odot (0 \ast y) = [(0 \odot y) \odot (0 \odot y) \ast (x \ast y) \odot (x \ast y) = 0 \odot (x \odot y)\).

(vi) For any \(x \in X\), by (P1) and (P2) we obtain \(0 \ast x = (x \odot x) \ast x = (x \ast x) \odot x = 0 \odot x\). \(\square\)
Theorem 3.6. For any pseudo BCH-algebra $X$, the set

$$K(X) := \{ x \in X \mid 0 \preceq x \}$$

is a pseudo subalgebra of $X$.

Proof. Let $x, y \in K(X)$. Then $0 \preceq x$ and $0 \preceq y$. Hence $0 \circ x = 0 \circ 0 = 0$ and $0 \circ y = 0 \circ 0 = 0$. Since $0 \circ (x \circ y) = (0 \circ x) \circ (0 \circ y) = 0 \circ 0 = 0$ and $0 \circ (x \circ y) = (0 \circ x) \circ (0 \circ y) = 0 \circ 0 = 0$, we have $x \circ y, x \circ y \in K(X)$. Thus $K(X)$ is a pseudo subalgebra of $X$. \qed

Proposition 3.7. Let $X$ be a pseudo BCH-algebra. If $x \in K(X)$ and $y \in X - K(X)$, then $x \circ y \in X - K(X)$ and $x \circ y \in X - K(X)$.

Proof. If $x \circ y \in K(X)$, then $x \circ (x \circ y) \in K(X)$ by Theorem 3.6. Using Proposition 3.5 (ii), we have $0 \preceq x \circ (x \circ y) \preceq y$ and so $y \in K(X)$. This is a contradiction. Now if $x \circ y \in K(X)$, then $x \circ (x \circ y) \in K(X)$ and so $0 \preceq x \circ (x \circ y) \preceq y$ by Proposition 3.5 (ii). Therefore $y \in K(X)$, which is a contradiction. \qed

Theorem 3.8. If $I$ is a pseudo ideal of a pseudo BCH-algebra $X$, then

(i) $\forall x, y, z \in X, x, y \in I, z \circ y \preceq x \Rightarrow z \in I$,

(ii) $\forall a, b, c \in X, a, b \in I, c \circ b \preceq a \Rightarrow c \in I$.

Proof. (i) Suppose that $I$ is a pseudo ideal of $X$ and let $x, y, z \in X$ be such that $x, y \in I$ and $z \circ y \preceq x$. Then $(z \circ y) \circ x = 0 \in I$. Since $x \in I$ and $I$ is a pseudo ideal of $X$, we have $z \circ y \in I$. Since $y \in I$ and $I$ is a pseudo ideal of $X$, we obtain $z \in I$. Thus (i) is valid.

(ii) Let $a, b, c \in X$ be such that $a, b \in I$ and $c \circ b \preceq a$. Then $(c \circ b) \circ a = 0 \in I$ and so $c \circ b \in I$. Since $b \in I$ and $I$ is a pseudo ideal of $X$, we have $c \in I$. Thus (ii) is true. \qed

Theorem 3.9. Let $I$ be a pseudo subalgebra of a pseudo BCH-algebra $X$. Then $I$ is a pseudo ideal of $X$ if and only if $\forall x, y \in X, x \in I, y \in X - I \Rightarrow y \circ x \in X - I$ and $y \circ x \in X - I$.

Proof. Assume that $I$ is a pseudo ideal of $X$ and let $x, y \in X$ be such that $x \in I$ and $y \in X - I$. If $y \circ x \notin X - I$, then $y \circ x \in I$. Since $I$ is a pseudo ideal of $X$, we have $y \in I$. This is a contradiction. Hence $y \circ x \in X - I$. Now if $y \circ x \notin X - I$, then $y \circ x \in I$ and so $y \in I$. This is a contradiction, and therefore $y \circ x \in X - I$.

Conversely, assume that $\forall x, y \in X, x \in I, y \in X - I \Rightarrow y \circ x \in X - I$ and $y \circ x \in X - I$. Since $I$ is a pseudo subalgebra, we have $0 \in I$. Let $x \in I, y \in X$ such that $y \circ x, y \circ x \in I$. If $y \notin I$, then $y \circ x, y \circ x \in X - I$ by assumption. This is a contradiction. Hence $y \in I$. Thus $I$ is a pseudo ideal of $X$. \qed
Proposition 3.10. Let $A$ be a pseudo ideal of a pseudo BCH-algebra $X$. If $B$ is a pseudo ideal of $A$, then it is a pseudo ideal of $X$.

Proof. Since $B$ is a pseudo ideal of $A$, we have $0 \in B$. Let $y, x \ast y, x \circ y \in B$ for some $x \in X$. If $x \in A$, then $x \in B$ since $B$ is a pseudo ideal of $A$. If $x \in X - A$, then $y, x \ast y, x \circ y \in B \subseteq A$ and so $x \in A$ because $A$ is a pseudo ideal of $X$. Thus $x \in B$ since $B$ is a pseudo ideal of $A$. This completes the proof.

Proposition 3.11. Let $I$ be a pseudo ideal of a pseudo BCH-algebra $X$. Then for all $x \in X$, $x \in X$ imply $0 \ast (0 \circ x) \in I$ and $0 \circ (0 \ast x) \in I$.

Proof. Let $x \in I$. Then $0 = (0 \circ x) \ast (0 \circ x) = (0 \ast (0 \circ x)) \circ x$ and $0 = (0 \ast x) \circ (0 \ast x) = (0 \circ (0 \ast x)) \ast x$ which imply that $0 \ast (0 \circ x), 0 \circ (0 \ast x) \in I$. This completes the proof.

Theorem 3.12. Let $I$ be a pseudo ideal of a pseudo BCH-algebra $X$ and let

$$I^\# := \{ x \in X | 0 \ast (0 \circ x), 0 \circ (0 \ast x) \in I \}.$$ 

Then $I^\#$ is a pseudo ideal of $X$ and $I \subseteq I^\#$.

Proof. Obviously, $0 \in I^\#$. Let $a \in X, y \in I^\#$ such that $a \ast y, a \circ y \in I^\#$. Then $0 \ast ((0 \circ (a \ast y)), 0 \circ (0 \ast (a \circ y)), 0 \ast ((0 \circ (a \circ y))) \in I$, and $0 \circ (0 \ast (a \circ y)) \in I$. Using Proposition 3.5 (v) and (vi), we have $(0 \ast (0 \circ a)) \ast (0 \circ (0 \ast y)) = 0 \circ ((0 \circ a) \circ (0 \ast y)) = 0 \circ (0 \ast (a \ast y)) \in I$ and $(0 \circ (0 \ast a)) \circ (0 \ast (0 \circ y)) = 0 \ast ((0 \ast a) \ast (0 \circ y)) = 0 \ast (0 \circ (a \circ y)) \in I$. Since $0 \ast (0 \circ y), 0 \circ (0 \ast y) \in I$, it follows from (PI2) that $0 \ast (0 \circ a), 0 \circ (0 \ast a) \in I$. Hence $a \in I^\#$. Thus $I^\#$ is a pseudo ideal of $X$. By Proposition 3.11, we know that $I \subseteq I^\#$. This completes the proof.

4 Pseudo atom

Definition 4.1. An element $a$ of a pseudo BCH-algebra $X$ is called a pseudo atom of $X$ if for every $x \in X$, $x \preceq a$ implies $x = a$.

Obviously, $0$ is a pseudo atom of $X$. Let $L(X)$ denote the set of all pseudo atoms of $X$, we call it the center of $X$.

Theorem 4.2. Let $X$ be a pseudo BCH-algebra. Then the following are equivalent: for all $x, y, z, w, u \in X$

(i) $w$ is a pseudo atom,

(ii) $w = x \circ (x \ast w)$ and $w = x \ast (x \circ w)$,

(iii) $(x \ast y) \circ (x \ast w) = w \ast y$ and $(x \circ y) \ast (x \circ w) = w \circ y$,
(iv) \( w \ast (x \odot y) = y \odot (x \ast w) \) and \( w \odot (x \ast y) = y \ast (x \odot w) \),
(v) \( 0 \odot (y \ast w) = w \ast y \) and \( 0 \ast (y \odot w) = w \odot y \),
(vi) \( 0 \odot (0 \ast w) = w \) and \( 0 \ast (0 \odot w) = w \),
(vii) \( 0 \odot (0 \ast (w \odot z)) = w \odot z \) and \( 0 \ast (0 \odot (w \ast z)) = w \ast z \),
(viii) \( z \odot (z \ast (w \odot u)) = w \odot u \) and \( z \ast (z \odot (w \ast u)) = w \ast u \).

**Proof.** (i) ⇒ (ii): Let \( w \) be a pseudo atom of \( X \). Since \( x \odot (x \ast w) \leq w \) and \( x \ast (x \odot w) \leq w \) by Proposition 3.5 (ii), we have \( w = x \odot (x \ast w) \) and \( w = x \ast (x \odot w) \).

(ii) ⇒ (iii): For every \( x \in X \), we obtain \( (x \odot y) \odot (x \ast w) = (x \odot (x \ast w)) \odot y = w \ast y \) and \( (x \odot y) \ast (x \odot w) = (x \ast (x \odot w)) \odot y = w \odot y \) by (P2) and (ii).

(iii) ⇒ (iv): Replacing \( y \) by \( x \odot y \) in (iii), we get \( w \ast (x \odot y) = (x \ast (x \odot y)) \odot (x \ast w) = (x \odot (x \ast w)) \odot (x \odot y) = y \odot (x \ast w) \) and \( w \odot (x \odot y) = (x \odot (x \odot y)) \ast (x \odot w) = (x \ast (x \odot w)) \odot (x \odot y) = y \ast (x \odot w) \).

(iv) ⇒ (v): Put \( y := 0 \) in (iv). Then \( w \ast (x \odot 0) = 0 \odot (x \ast w) \) and \( w \odot (x \ast 0) = 0 \ast (x \odot w) \). Hence \( w \ast x = 0 \odot (x \ast w) \) and \( w \odot x = 0 \ast (x \odot w) \) by Proposition 3.5 (iii).

(v) ⇒ (vi): Set \( y := 0 \) in (v). Then \( 0 \odot (0 \ast w) = w \ast 0 = w \) and \( 0 \ast (0 \odot w) = w \odot 0 = w \) by Proposition 3.5 (iii).

(vi) ⇒ (vii): For any \( w, z \in X \), we have \( 0 \odot (0 \ast (w \odot z)) = 0 \ast (0 \odot (w \odot z)) = 0 \ast (0 \odot (w \ast z)) = 0 \ast (0 \odot (0 \ast w)) \odot (0 \ast (0 \odot z)) = w \odot z \) and \( 0 \ast (0 \odot (w \odot z)) = 0 \ast (0 \odot (w \ast z)) = 0 \ast (0 \odot (0 \ast w)) \odot (0 \ast (0 \odot z)) = 0 \ast (0 \odot (0 \odot w)) \odot (0 \ast (0 \odot z)) = w \ast z \).

(vii) ⇒ (viii): For any \( u, w, z \in X \), we have \( w \odot u = 0 \odot (0 \ast (w \odot u)) = 0 \odot (z \odot (w \odot u)) = 0 \odot [(z \ast (w \odot u)) \odot z] = (0 \ast (z \ast (w \odot u))) \odot (0 \odot z) = (0 \odot (z \ast (w \odot u))) \odot (0 \odot z) = (0 \odot (0 \odot (z \odot 0))) \odot (z \ast (w \odot u)) = (0 \odot (0 \odot (z \odot 0))) \odot (z \ast (w \odot u)) = (z \odot 0) \odot (z \ast (w \odot u)) = z \odot (z \ast (w \odot u)). \) By a similar way, we obtain \( z \ast (z \odot (w \odot u)) = w \ast u \).

(viii) ⇒ (i): If \( z \ast x = z \odot x = 0 \), then by (viii) we have \( x = x \odot 0 = z \ast (z \odot (x \ast 0)) = z \ast (z \odot x) = z \odot 0 = z \). This shows that \( x \) is a pseudo atom of \( X \). This completes the proof.

**Corollary 4.3.** Let \( X \) be a pseudo \( BCH \)-algebra. If \( a \) is a pseudo atom of \( X \), then for all \( x \in X \), \( a \ast x \) and \( a \odot x \) are pseudo atoms. Hence \( L(X) \) is a pseudo subalgebra of \( X \).

**Corollary 4.4.** Let \( X \) be a pseudo \( BCH \)-algebra. For every \( x \in X \), there is a pseudo atom \( a \) such that \( a \leq x \), i.e., every pseudo \( BCH \)-algebra is generated by a pseudo atom.

**Proposition 4.5.** A non-zero element \( a \in X \) is a pseudo atom of a pseudo \( BCH \)-algebra \( X \) if \( \{0, a\} \) is a pseudo ideal of \( X \).
Proof. Let \( x \leq a \) for any \( x \in X \). Then \( x * a = x \diamond a = 0 \in \{0, a\} \). Since \( x \in \{0, a\} \) is a pseudo ideal of \( X \), we have \( x = 0 \) or \( x = a \). Since \( a \neq 0 \), we obtain \( x = a \). Hence \( a \) is a pseudo atom of \( X \).

**Proposition 4.6.** If non-zero element of a pseudo BCH-algebra \( X \) is a pseudo atom, then any pseudo subalgebra of \( X \) is a pseudo ideal of \( X \).

Proof. Let \( S \) be a pseudo subalgebra of \( X \) and let \( x, y \in S \). By Theorem 4.2, we have \( y = x * (x \diamond y) = x * ((0 * (y \diamond x)) \diamond x) = 0 \). Thus any pseudo subalgebra of \( X \) is a pseudo ideal of \( X \).

For pseudo atom \( a \) of a Pseudo BCH-algebra \( X \), \( V(a) := \{ x \in X | x \leq a \} \) is called a pseudo branch of \( X \).

**Theorem 4.7.** Let \( X \) be a pseudo BCH-algebra. Suppose that \( a \) and \( b \) are pseudo atoms of \( X \). Then the following properties hold:

(i) For all \( x \in V(a) \) and all \( y \in V(b) \), \( x * y \in V(a * b) \) and \( x \diamond y \in V(a \diamond b) \).

(ii) For all \( x \) and \( y \in V(a) \), \( x \diamond y, x \ast y \in K(X) \), where \( K(X) = \{ x \in X | 0 \leq x \} \).

(iii) If \( a \neq b \), then for all \( x \in V(a) \) and \( y \in V(b) \), we have \( x \ast y, x \diamond y \in K(X) \).

(iv) For all \( x \in V(b) \), \( a \ast x = a \ast b \) and \( a \diamond x = a \diamond b \).

(v) If \( a \neq b \), then \( V(a) \cap V(b) = \emptyset \).

Proof. (i) For all \( x \in V(a) \) and all \( y \in V(b) \), by Proposition 3.5 and Theorem 4.2 we have \( (a \ast b) \circ (x \ast y) = (a \ast ((0 \circ (a \ast b))) \circ x \ast y) = (a \ast ((0 \circ (a \ast b))) \circ x) \ast y = ((a \ast b) \circ x) \ast y = ((a \ast b) \circ x) \ast y = ((a \ast b) \circ x) \ast y = ((a \ast b) \circ x) \ast y = 0 \). Moreover, \( (a \circ b) \circ (x \circ y) = (a \circ (b \circ x)) \circ y = (a \circ (b \circ x)) \circ y = (a \circ (b \circ x)) \circ y = 0 \). Hence \( x \ast y \in V(a \ast b) \) and \( x \circ y \in V(a \circ b) \).

(ii) and (iii) are simple consequences of (i).

(iv) For all \( x \in V(b) \), by Theorem 4.2 we have \( (a \ast x) \circ (a \ast b) = (a \circ ((a \ast b))) \ast x = b \ast x = 0 \). Moreover, \( a \ast b \) is a pseudo atom by Corollary 4.3. Therefore \( a \ast x = a \ast b \). Also we get \( (a \circ x) \circ (a \circ b) = ((a \ast (a \circ b))) \ast x = b \ast x = 0 \). Moreover, \( a \circ b \) is a pseudo atom by Corollary 4.3. Therefore \( a \circ x = a \circ b \).

(v) Let \( a \neq b \) and \( V(a) \cap V(b) \neq \emptyset \). Then there exists \( c \in V(a) \cap V(b) \). By (i), we have \( 0 = c \ast c = c \circ c \in V(a \ast b), V(a \circ b) \). Hence \( a \ast b = a \circ b = 0 \), which is a contradiction. Thus (v) is true.
References


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