Some Results in Fredholm Theory via The Measure of Noncompactness

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Abstract

Let $A$ be a bounded linear operator in a complex Banach space $X$. We show that $\text{Id}_X - A$ is a Fredholm operator provided that $A$ has a sufficiently small polynomially measure of noncompactness. In our general framework, we note that the case of Riesz operator becomes a particular one as it is for the other results in the domain. This enable us to obtain a new characterization for the Weyl essential spectrum of a closed densely defined operators.

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1 Introduction

Let $\mathcal{L}(X)$ be the algebra of all bounded linear operators on a complex Banach space $X$. The notion of a measure of noncompactness turns out to be an
useful tool in applications of mathematical analysis especially in the study of solutions of nonlinear and integral equations, ordinary and partial differential equations, the fixed point theory and in the characterization of compact linear operators in Banach spaces (cf. [1], [2], [4], [15]). To show the power of this concept in real life, some applications to crystallography and physics can be found, for instance, in (cf. [10], [11]).

The aim of this article is to establish, via the measure of noncompactness, the most conditions that can be satisfied by a bounded linear operator to resolve the so-called Fredholm alternative equation (cf. [1], [15], [16]).

**Definition 1.1** The operator $A \in \mathcal{L}(X)$ is said to be compact, if its domain is all of $X$ and, for every bounded sequence $(x_n)_n$ in $X$, the sequence $(A(x_n))_n$ has a convergent subsequence in $X$. We denote by $\mathcal{K}(X)$ the ideal, of the algebra $\mathcal{L}(X)$, of all compact operators on $X$.

**Remark 1.2** It is sufficient to compactness that the image of the closed unit ball should be relatively compact.

One of the principal consequences of the assumption that $A$ is compact is that it is possible to characterize its spectrum $\sigma(A)$ in a particularly simple manner. Indeed, it is already known that every non-zero point of $\sigma(A)$ is an eigenvalue, and it will be proved that there are only a finite number of points in any region excluding a neighborhood of the origin.

For $A \in \mathcal{L}(X)$, the sets $\sigma(A)$, $\rho(A)$, $N(A)$ and $R(A)$ designed, respectively, the spectrum, the resolvent set, the null space, and the range of $A$. We set $\alpha(A) = \dim[N(A)]$ and $\beta(A) = \text{codim}[R(A)]$.

**Definition 1.3** An operator $A \in \mathcal{L}(X)$ is a Fredholm operator if both $\alpha(A)$ and $\beta(A)$ are finite and $R(A)$ is closed in $X$. The index of the operator $A$ is the number $i(A) = \alpha(A) - \beta(A)$. The sets of upper and lower semi-Fredholm operators are defined as

$$
\mathcal{F}_+(X) = \{A \in \mathcal{L}(X) : \alpha(A) < \infty, \text{ } R(A) \text{ is closed in } X \}
$$

$$
\mathcal{F}_-(X) = \{A \in \mathcal{L}(X) : \beta(A) < \infty, \text{then } R(A) \text{ is closed in } X \}
$$

and the set of Fredholm operators on $X$ is defined as $\mathcal{F}(X) = \mathcal{F}_+(X) \cap \mathcal{F}_-(X)$.

For a systematic treatment of the Fredholm operators and a good account of the theory see (cf. [1], [16]).

**Definition 1.4** Let $B_1(X)$ be the unit ball in the Banach space $X$, the measure of noncompactness, commonly called the Hausdorff MNC, of a bounded set $Y \subset X$ is

$$
\delta(Y) = \inf \{\delta > 0 : \exists \text{ a finite subset } M \text{ of } X \text{ such that } Y \subset M + \delta B_1(X)\}.
$$
It is clear that $\delta(Y) = 0$ if and only if $Y$ is a relatively compact subset of $X$. Moreover, for $T \in \mathcal{L}(X)$, the measure of noncompactness of $T$ is defined as being the number

$$\delta(T) = \delta[T(B_1(X))] .$$

The notion of measure of noncompactness of bounded operators bring us a sufficient conditions for which $\text{Id}_X - A \in \mathcal{F}(X)$. Indeed, in the second section, we show that if $A \in \mathcal{L}(X)$ and $a_0, a_1, \cdots, a_n \in \mathbb{C}$ with $a_0 \neq 0$ such that

$$\delta \left[ \frac{1}{a_0} \left[ (a_0 - a_1)A + (a_1 - a_2)A^2 + \cdots + a_n A^{n+1} \right] \right] < \frac{1}{2}$$

then $\text{Id}_X - A \in \mathcal{F}(X)$.

The third section is devoted to the Riesz operators. Moreover, an extension of the analysis carried out by K. Latrach and A. Dehici (cf. [9]) was established and finally a new characterization for the Weyl essential spectrum of closed densely defined operators is given.

# 2 Main Results

We give a survey of the basic properties, needed in this paper, that satisfied by the measure of noncompactness.

**Lemma 2.1** Let $X$ a Banach space and $\delta$ a measure of noncompactness defined on it :

1. If $Z \subset Y$ then $\delta(Z) \leq \delta(Y)$
2. If $A \in \mathcal{L}(X)$ then $\delta$ is a semi-norm on $\mathcal{L}(X)$. Moreover, $\delta(A) \leq \|A\|$.
3. If $A, B \in \mathcal{L}(X)$ then $\delta(AB) \leq \delta(A)\delta(B)$.
4. If $K \in \mathcal{K}(X)$ and $Y \subset X$ then $\delta(Y + K) = \delta(Y)$.
5. We have $\frac{1}{2}\delta(A^*) \leq \delta(A) \leq 2\delta(A^*)$ where $A^*$ denotes the dual operator of $A \in \mathcal{L}(X)$.

**Proof.** See (cf. [4]) for the proof and most properties of the measure $\delta$. $\square$

Our main result is the following theorem :

**Theorem 2.2** Let $X$ be a Banach space and $A \in \mathcal{L}(X)$. Assume that there exists $a_1, a_2, \cdots, a_n \in \mathbb{C}$ and $a_0 \in \mathbb{C}^*$

1. If

$$\delta \left[ \sum_{j=0}^{n-1} \left( \frac{a_j - a_{j+1}}{a_0} \right) A^{j+1} + \left( \frac{a_n}{a_0} A^{n+1} \right) \right] < 1$$

then $\text{Id}_X - A \in \mathcal{F}_+(X)$.
Moreover, if

$$\delta \left[ \sum_{j=0}^{n-1} \left( \frac{a_j - a_{j+1}}{a_0} \right) A^{j+1} + \left( \frac{a_n}{a_0} A^{n+1} \right) \right] < \frac{1}{2}$$

then \( \text{Id}_X - A \in \mathcal{F}(X) \).

**Proof.** Put \( T = \text{Id}_X - A \).

1. We claim that \( \alpha(T) < \infty \). For this end, we will prove that the set \( N(T) \cap B_1(X) \) is a compact one. More precisely, we will show that if \( Z \) is a compact subset of \( X \) then the set \( Y = \{ x \in B_1(X) : T(x) \in Z \} \) is empty or a compact subset of \( X \). Indeed, let assume that \( Y \) is not empty. Fix \( z \in M \) and \( x \in B_1(X) \) with \( Tx = z \). By applying the operator \( a_0A^0 + a_1A + \cdots + a_nA^n \) to the equation \( Tx = z \), we obtain

$$x = \sum_{j=0}^{n-1} \left( \frac{a_j - a_{j+1}}{a_0} \right) A^{j+1}(x) + \frac{a_n}{a_0} A^{n+1}(x) + \sum_{j=0}^{n} \frac{a_j}{a_0} A^j(z).$$

Since the range of a compact set by a bounded operator is compact, it follows that:

$$\tilde{Z} = \left\{ \sum_{j=0}^{n} \frac{a_j}{a_0} A^j(z) : z \in Z \right\}$$

is a compact set of \( X \). Put

$$U = \sum_{j=0}^{n-1} \left( \frac{a_j - a_{j+1}}{a_0} \right) A^{j+1} + \frac{a_n}{a_0} A^{n+1}.$$ 

Then it is clear that \( Y \subseteq UY + \tilde{Z} \). Then using the assertion 1) of lemma 2.1, it comes that

$$\delta(Y) \leq \delta(UY + \tilde{Z}) \leq \delta(UY) \leq \delta(U) \delta(Y).$$

By assumption we have \( \delta(U) < 1 \), thus \( \delta(Y) = 0 \) which shows that \( Y \) is a compact set in \( X \). To establish the result dealing with the set \( N(T) \cap B_1(X) \), it suffices that \( Z = \{0\} \).

To complete the proof, we will verify the closedness of the range \( R(T) \) in \( X \). Since \( N(T) \) is a finite dimensional space, then there exists a closed subspace \( F \) of \( X \) such that \( X = N(T) \oplus F \). Now we will establish the inequality:

$$c ||T(x)|| \geq ||x|| \quad \text{for all } x \in F \text{ and with } c > 0.$$
Suppose the converse is satisfied, then for all \( n \in \mathbb{N} \), there exists \( x_n \in F \) with norm \( 1 \) satisfying \(||T(x_n)||| \leq \frac{1}{n}\), it follow that \( T(x_n) \to 0 \) when \( n \to +\infty \). Afterwards, by taking

\[
M = \{T(x_n) : n \in \mathbb{N}\} \bigcup \{0\}
\]

and according to the first part, it follows that the sequence \( \{x_n\}_n \) admits a subsequence \( \{x_{nk}\}_n \) which converge to \( x_0 \in Y \). In addition, it is easy to see that \( c > 0 \) such that \( c||T(x)|| \geq ||x|| \) which establish the closedness of the range \( R(T) \).

2. Assume that we have

\[
\delta \left[ \sum_{j=0}^{n-1} \left( \frac{a_j - a_{j+1}}{a_0} \right) A^{j+1} + \left( \frac{a_n}{a_0} A^{n+1} \right) \right] < \frac{1}{2}.
\]

The assertion 5) in lemma 2.1 shows that \( \delta(U^*) < 2\delta(U) < 1 \) where \( U^* \) denotes the dual operator of \( U \). With the same process as in 1), one obtain \( \alpha(U^*) = \beta(U) < \infty \). So \( T^* = \text{Id}_X^* - A^* \in \mathcal{F}_+(X^*) \) thus \( T = \text{Id}_X - A \in \mathcal{F}_-(X) \) and consequently \( T \in \mathcal{F}_-(X) \bigcap \mathcal{F}_+(X) = \mathcal{F}(X) \). \( \square \)

We will discuss the index of the operator \( T = \text{Id}_X - A \) via the measure of noncompactness of the operator \( A \).

**Theorem 2.3** Let \( X \) be a Banach space and \( A \in \mathcal{L}(X) \)

1. If \( \delta(A) < 1 \), then \( T = \text{Id}_X - A \) is a Fredholm operator with index 0.

2. Let assume that \( \delta(A) \geq 1 \) and there exists \( a_1, a_2, \cdots, a_n \in \mathbb{C} \) and \( a_0 \in \mathbb{C}^* \) such that

\[
\delta \left[ \sum_{j=0}^{n-1} \left( \frac{a_j - a_{j+1}}{a_0} \right) A^{j+1} + \left( \frac{a_n}{a_0} A^{n+1} \right) \right] < \frac{1}{2}
\]

and

\[
\max \left[ \max_{0 \leq j \leq n-1} \left| \frac{a_j - a_{j+1}}{a_0} \right| , \left| \frac{a_n}{a_0} \right| \right] < \frac{1}{2(n+1)\delta(A)^{n+1}}
\]

then \( T = \text{Id}_X - A \in \mathcal{F}(X) \) and \( i(T) = 0 \).

**Proof.** According to the assumptions on the measure \( \delta(A) \):

1. It follows that \( \lim_{k \to +\infty} \delta(A)^k = 0 \). So, there exists \( k_0 \in \mathbb{N}^* \) such that \( \delta(A)^{k+1} < \frac{1}{2} \) for all \( k \geq k_0 \). In addition, assertion 2) of lemma 2.1 asserts that \( \delta(A^{k+1}) < \delta(A)^{k+1} < \frac{1}{2} \), by theorem 2.2 with \( a_0 = a_1 = \cdots = a_n = 1 \) and
\[
\cdots = a_0 = 1, \text{ it follows that } T \in \mathcal{F}(X). \text{ Moreover, for all } t \in [0,1], \text{ we have } [\delta(tA)]^{k+1} < \frac{1}{2}, \text{ thus } \text{Id}_X - tA \in \mathcal{F}(X). \text{ On the other hand, the fact that the index is constant in each connected component of } \mathcal{F}(X) \text{ and the compactness of } [0,1] \text{ imply that } i(T) = i(\text{Id}_X - tA) = i(\text{Id}_X) = 0.
\]

2. Let \( t \in [0,1], \) introduce the application \( h_t \) given by
\[
h_t(A) = \delta \left[ \sum_{j=0}^{n-1} \left( \frac{a_j - a_{j+1}}{a_0} \right) (tA)^{j+1} + \left( \frac{a_n}{a_0} \right) (tA)^{n+1} \right].
\]

A sample calculation shows that
\[
h_t(A) < \left( \sum_{j=0}^{n-1} \left| \frac{a_j - a_{j+1}}{a_0} \right| \delta(A^{j+1}) + \left| \frac{a_n}{a_0} \right| t^{n+1}\delta(A^{n+1}) \right).
\]
\[
< \left( \sum_{j=0}^{n-1} \left| \frac{a_j - a_{j+1}}{a_0} \right| + \left| \frac{a_n}{a_0} \right| \right) \left[ \delta(A) \right]^{n+1}.
\]
\[
< (n+1) \max \left[ \max_{0 \leq j \leq n-1} \left| \frac{a_j - a_{j+1}}{a_0} \right|, \left| \frac{a_n}{a_0} \right| \right] \left[ \delta(A) \right]^{n+1}.
\]

Therefore, by using the theorem assumptions, we deduce that \( \text{Id}_X - tA \in \mathcal{F}(X) \) for all \( t \in [0,1] \). Now, while proceeding by the same way as in the proof of 1), we obtain \( i(T) = i(\text{Id}_X - tA) = i(T) \). \( \square \)

**Remark 2.4** Let \( P(z) = b_1z + b_2z^2 + \cdots + b_nz^n \) a polynomial with complex coefficients and let \( A \) a bounded operator on \( X \) verifying the inequality \( \delta(b_1A + b_2A^2 + \cdots + b_nA^n) \). An essential condition ensuring that \( \text{Id}_X - A \in \mathcal{F}(X) \) is that \( b_1 + b_2 + \cdots + b_n = 1 \). In other words, this is equivalent to find numbers \( a_0, a_1, \cdots, a_n \in \mathbb{C} \) with \( a_0 \neq 0 \) which are solutions of the following system
\[
\begin{align*}
\frac{a_0 - a_1}{a_0} &= b_1 \\
\frac{a_1 - a_2}{a_0} &= b_2 \\
\vdots \\
\frac{a_n}{a_0} &= b_n.
\end{align*}
\]

**Counterexample** Let \( X \) be a decomposable Banach space under the direct sum \( X = X_1 \oplus X_2 \). We denote by \( A \) the projection on \( X_1 \) and let \( P(z) = z^2 - z \). It is easy to observe that \( A^2 - A = 0 \) then \( P(A) = 0 \) and \( \delta(A^2 - A) = 0 < \frac{1}{2} \), however \( \text{Id}_X - A \notin \mathcal{F}(X) \) because \( N(\text{Id}_X - A) = X_1 \) is an infinite dimensional closed subspace (here \( b_1 + b_2 \neq 1 \)).
3 The framework of Riesz operators

By using the concept of measure of noncompactness, we will show that many well known results in the area are particular cases in our general framework. First, Let us start by the definition of Riesz operator.

**Definition 3.1** Let $X$ a Banach space and let $R \in \mathcal{L}(X)$. The operator $R$ is a Riesz operator if $R$ satisfies the Riesz-Schauder theory of compact operators. In other words, if the spectrum $\sigma(R)$ is made up at most of a sequence of eigenvalues of finite algebraic multiplicities which accumulate to 0.

**Remark 3.2** Denote by $\mathcal{R}(X)$ the subfamily of $\mathcal{L}(X)$ of Riesz operators.

- A key question to give a characterization of $\mathcal{R}(X)$ is to use those of compact and quasinilpotant operators.
- One of the problems involved in our subject is the famous West decomposition which has a positive answer only in the separable Hilbert and $L^p$ ($1 < p < +\infty$) spaces (cf. [3], [18]).

**Proposition 3.3** Let $X$ be a Banach space and let $R \in \mathcal{R}(X)$, then for all $1 > \varepsilon > 0$, there exists $n(\varepsilon) \geq 1$ such that

$$\delta(R^n) \leq \varepsilon.$$

**Proof.** The fact that $R \in \mathcal{R}(X)$ satisfies the Riesz-Schauder theory shows that the set of points $\lambda \in \sigma(R)$ which satisfies the inequality $|\lambda| > \varepsilon$ consists of finite number $\{\lambda_1, \lambda_2, \cdots, \lambda_m\}$. Let $P_i$ be the projection of $X$ on the spaces $N(\lambda_i - R)$ in the decomposition

$$X = N(\lambda_i - R) \bigoplus H(\lambda_i - R) \quad 1 \leq i \leq m_i.$$

Introduce a new operator $V$ as

$$V = R - \sum_{i=1}^{m_i} R \circ P_i.$$

It is easy to see that $\sum_{i=1}^{m_i} R \circ P_i$ is a finite rank operator thus compact. Moreover, we have

$$r_p(V) = \lim_{n \to +\infty} \|V^n\|^{\frac{1}{n}}$$

which shows the existence of an integer $n_\varepsilon$ such that $\|V^n\| \leq \varepsilon$. On the other hand, we can write $R^n = V^n + F_{n_\varepsilon}$ where $F_{n_\varepsilon}$ is a compact operator which implies that

$$\delta(R^n) = \delta(V^n) \leq \varepsilon. \quad \square$$

As a consequence of this result, we have the following classical property of Riesz operators:
Proposition 3.4 Let $X$ be a Banach space and $A \in \mathcal{R}(X)$, then
\[ \forall \lambda \in \mathbb{C}, \quad \lambda \text{Id}_X - A \in \mathcal{F}(X) \quad \text{and} \quad i(\text{Id}_X - A) = 0. \]

Proof. For the case $|\lambda| \geq 1$, we use the decomposition $\lambda \text{Id}_X - A = \lambda (\text{Id}_X - \frac{1}{\lambda} A)$. In addition, by applying the proposition 3.3 to the operator $\frac{1}{\lambda} A$ by taking $\varepsilon = \frac{1}{2}$ we note that $\text{Id}_X - \frac{1}{\lambda} A \in \mathcal{F}(X)$. The case $\lambda \in D(0, 1) \setminus \{0\}$ is done by using the fact that the set $\mathcal{F}(X)$ is open in $L(X)$ (equipped with the norm topology), step by step, we deduce that for all $\varepsilon > 0$ we have $\lambda \text{Id}_X - A \in \mathcal{F}(X)$ for all $\lambda \in \{\mu \in \mathbb{C}, \varepsilon \leq |\mu| \leq 1\}$ this implies that $\lambda \text{Id}_X - A \in \mathcal{F}(X)$ for all $\lambda \in D(0, 1) \setminus \{0\}$. On the other hand, the connexity of the set $\mathbb{C}^*$, the fact that $\text{Id}_X - \frac{1}{\lambda} A$ is a bijection for $|\lambda|$ sufficiently big together with the stability of the index on the connected components of the set $\mathcal{F}(X)_A = \{\lambda \in \mathbb{C} : \lambda \text{Id}_X - A \in \mathcal{F}(X)\}$ show that $i(\lambda \text{Id}_X - A) = 0$ for all $\lambda \in C^*$. □

Definition 3.5 Let $X$ be a Banach space and $A$ a closed densely defined operator on $X$. We denote by $\sigma_\omega(A)$ the Weyl essential spectrum of $A$ which is defined by
\[ \sigma_\omega(A) = \bigcap_{K \in \mathcal{K}(X)} \sigma(A + K). \]

Let us note by $\mathcal{I}_P(X)$ the greatest closed two-sided ideal included in $\mathcal{R}(X)$. This ideal also known as Fredholm Perturbations (cf. [8], [9]). In (cf. [9]), the authors has established a characterization of the Weyl essential spectrum by using the class of operators $\mathcal{I}_P(X)$. More precisely, they showed that
\[ \sigma_\omega(A) = \bigcap_{F \in \mathcal{I}_P(X)} \sigma(A + F). \]

Let $A$ be a closed densely defined operator on $X$. We define the sets $\mathcal{M}(X)$ as being the operators $M \in L(X)$ that verifying the results of theorem 2.2 and
\[ \mathcal{S}_A(X) = \{S \in \mathcal{L}(X) : (-\lambda \text{Id}_X + A + S)^{-1} S \in \mathcal{M}(X) \quad \text{forall} \quad \lambda \in \rho(A + S)\}. \]

The following characterization of the Weyl essential spectrum $\sigma_\omega(A)$ extend the results established in (cf. [9]).

Theorem 3.6 Let $A$ be a closed densely defined operator on $X$, then we have
\[ \sigma_\omega(A) = \bigcap_{S \in \mathcal{S}_A(X)} \sigma(A + S). \]
Proof. It is clear that \( \bigcap_{S \in S_A(X)} \sigma(A+S) \subseteq \sigma_\omega(A) \) since \( K(X) \subseteq S_A(X) \). For the opposite inclusion, consider \( \lambda \in \sigma_\omega(A) \) and assume that \( \lambda \notin \bigcap_{S \in S_A(X)} \sigma(A+S) \). This implies the existence of an operator \( S_0 \in S_A(X) \) such that \( \lambda \notin \sigma(A+S_0) \). On the other hand, we can write
\[
\lambda \Id_{X} - A[(\lambda \Id_{X} + A + S_0)^{-1}S_0](\lambda \Id_{X} - A - S_0).
\]
The fact that \( [(\lambda \Id_{X} + A + S_0)^{-1}S_0] \in \mathcal{M}(X) \) shows that
\[
\Id_{X} + (\lambda \Id_{X} + A + S_0)^{-1}S_0 \in \mathcal{F}(X) \quad \text{and} \quad i(\lambda \Id_{X} + A + S_0)^{-1}S_0) = 0.
\]
Afterwards, by using (cf. [15, theorem 1.3 p. 163]), we obtain
\[
\lambda \Id_{X} - A = (\lambda \Id_{X} - A - S_0)[\Id_{X} + (\lambda \Id_{X} - A - S_0)^{-1}S_0] \in \mathcal{F}(X) \quad \text{and} \quad i(\lambda \Id_{X} - A) = 0.
\]
This is a contradiction, hence \( \sigma_\omega(A) \subseteq \bigcap_{S \in S_A(X)} \sigma(A+S) \). \( \square \)

Remark 3.7 According to the proposition 3.3, we see that \( \mathcal{I}_P(X) \subseteq S_A(X) \), from that, we note that the theorem 3.6 represents an extension of (cf. [9, theorem 3.4]). Moreover, we can replace \( S_A(X) \) by any class \( \mathcal{I}_A \subseteq S_A(X) \) in theorem 3.6.

References


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