

# **Dynamic Consequences of Lotka-Volterra Predation Model when the Area Inhabited by the Preys and Predators Decreases**

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## **Abstract**

In this work, we use an analytical approach to study the dynamic consequences of Lotka-Volterra predation model when the area of where inhabit the preys and predators decreases. We demonstrate that loses temper the food chain and its stability, although they take measured proofreaders to maintain environmental carrying capacity.

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## 1 Introduction

The Lotka-Volterra predator prey model common is

$$\begin{aligned}\frac{dX}{dt} &= r\left(1 - \frac{X}{K}\right) X - qXY \\ \frac{dY}{dt} &= (pX - c) Y\end{aligned}\tag{1}$$

The parameters have the following biological meanings:

$r$ : is the intrinsic per capita prey growth rate;

$K$ : is the prey environmental carrying capacity;

$q$ : is the maximal per capita predator consumption rate;

$p$ : is the efficiency with which predators convert consumed prey into new predators;

$c$ : is the natural per capita death predator rate.

As  $q = \sigma \frac{S_y}{S}$  where  $\sigma$  it is the fraction of encounters prey-predator where the prey dies.  $S_y$  and  $S$  it is the area where each predator looks for the preys and it is the region where the preys are distributed respectively. And  $p = \xi q$  where  $\xi$  it is the quantity of new predators taken place by each consumed prey.

If we diminish the area, let us say that it exists  $0 < \phi < 1$  such that the not intervened area is  $\phi S < S$ , the consumption average changes to  $\frac{1}{\phi}q > q$ , then the rate of conversion of preys in new predators rate changes to  $\frac{1}{\phi}p > p$ . We want to see the behavior of the equilibrium point where the species coexist.

If we want to keep a healthy life and to enjoy the marvels of the nature, an appropriate handling of the ecosystems should be had. The image of the nature like an inexhaustible source of wealth has remained per years and only recently we have taken minimum conscience of the impact of the excessive extraction of its resources. An appropriate handling of the natural resources should be made. When people modify an ecosystem to obtain something, this, in compensation, usually causes negative effects on other components of the same one. The human from the beginning of the humanity is extracting renewable and not renewable resources of the nature to subsist and to accumulate wealth, producing salification, erosion, compaction, desertification, decrease of the biodiversity, monocultures, deforestation, contamination in the ecosystems. However, the conservation or the improvement of certain components of an ecosystem, can take to positive synergies.

Human began to settle in towns about 10.000 years ago, maybe the maximum quantity of people was few millions, a number that didn't affect from an important way to the ecosystem of the planet. The big advances of the scientific

knowledge in the agriculture, the industry, the medicine and the social organization made possible the considerable growth of the population. In the same measure that has gone growing the number of population in the planet, the levels of the environmental deterioration have been increased and with them, they have diminished the levels of drinkable water, they have gotten lost different vegetable and animal species, numerous incurable illnesses have appeared, have been deteriorated the floor, which is the support where the necessary foods take place so that the human species exist and persist. The population density is proportional to the impact on the surface of the planet and also to the lot of civil works constructed; the civil works modify the natural landscape, and therefore the ecosystem.

The constructions of civil works that look for to improve the level of the man's life, impact negatively the ecosystems, because when we cut off areas of the ecosystems we lose the temper of the food chain that maintains the stability of the same one, and consequently their carrying and capacity, putting in danger the organisms that live in them. These and other realities of our planet are phenomena sufficiently serious to alert the human society to do the necessary means to break the deterioration. The high demographic density that produces big urban establishments and a quick urban growth (many times without planning), give place to situations of accumulation or an inappropriate sanitary infrastructures for the waste handle, dilute residual and contamination for the wrong handling of the residuals, generating a rupture of the primitive ecological balance. The problems of atmospheric contamination are closely bound to the use of the energy and bring problems of health for the population; damages to the buildings and the vegetation, and an important source of emission of gases of effect hothouse whose consequences are already feel at global level.

A forest is not a simply meeting of trees that covers a territory, but rather it constitutes a biological community, a group of organisms that are sustained with base on relationships structured by the nature along thousands or millions of years. With the result that any human intervention not well drifted on an arboreal community, it causes disturbances in the whole biological engagement. The humanity has transformed the environment and the primitive ecosystems progressively, represented by wide wooded surfaces, to do this; they have used different technical and methods of exploitation.

The central topic of discussion for several years in the civil engineering and on which bigger attention is lent is the effect of the man's works on the nature (human action). The evaluation of the environmental impact (EEI) of the civil works, allows you to take actions to minimize and/or to annul the possible environmental consequences of the projects, and to take measures to reduce, correct and compensate. A very important aspect of the corrective measures is the cost of the same ones, since this cost is not marginal regarding of the

substantive work and it can produce strong increments in the total value of the project, for what it is important to consider it as soon as possible. But it is not only important to keep in mind the viability from the economic point of view but also the technique, the effectiveness (to reduce the impact), the efficiency (Cost/Impact), the installation easiness and maintenance and control (since usually the measures, once implanted give way).

The fundamental objectives of any EEI are:

1. To describe and to analyze the project (as much in their contents as in their objectives), since it is the interference that will generate the impact.
2. To define and to value the means on which will have effects the project, since the objective of an Evaluation of the Environmental Impact consists on to minimize and/or to annul the possible environmental consequences of the projects.
3. To foresee the generated environmental effects and to evaluate them to be able to judge the suitability of the work, as well as to allow, or not, their realization under the best possible conditions of environmental sustainability.
4. To identify measures for reducing, correct and compensate.

It is around this last one objective that the present work is developed.

***Applications of mathematical models.*** With the application of models, the equilibrium is looked between conservation and exploitation, adopting intervention methods that don't destroy the potential of recovery of an ecosystem. The idea is to prevent the loss of its productive capacity and its genetic diversity, to assure the sustainability, keeping in mind that in general terms, activities like the construction of civil works, to improve the level of the man's life, impacts the environment because intervening the ecosystems and clipping its areas, loses temper the food chain that maintains the stability of the same one, and its carrying capacity.

Population ecology has given emphasis on the introduction of natural complexity and realism into the basic Lotka-Volterra framework. The initial steps were to include density-dependent effects on the endogenous dynamic of predators and preys, and to develop non-linear functions for consumption of prey by predators, the so-called functional response [2, 4, 5, 6]. In this way, the predator behavior was explicitly considered in predator-prey models.

More recently, the behavior of prey and its consequences at the population level has been worked out and incorporated into the predation theory [1] or in the growth prey function due the Allee effect [3].

In this work, it is shown mathematically up to where you can intervene the area of an ecosystem when a refuge is included, taking correctives to maintain

the load capacity in the system, without modifying the stability of the same one, taking a section of the food chain and modeling the pillaging relationship, it is determined how it affects the area of the construction lot from a civil work to an ecosystem.

The dynamics of the system prey predator has been studied extensively in this century, the effects of the pillaging in the population's dynamics are diverse and they extend from the extinction of the prey with a subsequent extinction of the predator until the coexistence of both in the population in certain balance. Fundamentally, the interaction predator-prey is described by means of autonomous systems of differential equations of order two, in particular the so called Gause, Logistical or Kolmogorov type models.

The models proposed for interactions predator-prey have considered diverse suppositions to simplify their mathematical descriptions, such as: The populations' homogeneity, homogeneity of environmental, uniform spatial distribution, constant rates of growth, encounters between the species predators and equally probable prey, sizes population clerks exclusively of the time, the species predators feeds exclusively of the species prey, while this feeds of a resource that is in the habitat in big quantities the one which alone it intervenes passively, they are not considered behaviors of the species of physiologic, morphological, social type, neither reintroduction of species etc.

We denote by  $X(t) = X$  and  $Y(t) = Y$  the population sizes of preys and predators, respectively for  $t > 0$ , considered as continuous variables that can represent density, biomass or quantity of each population's individuals.

The equilibrium points of the Lotka-Volterra predator prey models (1) are:

$(0, 0)$  is hyperbolic saddle for all parameter values.

$(K, 0)$  is hyperbolic saddle for  $pK - c > 0$  an attractor equilibrium point for  $pK - c < 0$  and  $(X_o^*, Y_o^*) = (\frac{c}{p}, \frac{r}{q}(\frac{pK-c}{pK}))$  exists and is globally asymptotically stable if  $K > \frac{c}{p}$ . (The state of coexistence is alone feasible if the predator at least can manage some production excess when the prey is in her carrying capacity).

The global stability is proved using the Liapunov function

$$V(X, Y) = c_1(\frac{p}{c}X - (1 + \ln \frac{p}{c}X)) + c_2(\frac{pqK}{r(pK - c)}Y - (1 + \ln \frac{pqK}{r(pK - c)}Y))$$

## 2 Models when we clip the area where the species live.

As  $q = \sigma \frac{S_y}{S}$  where  $\sigma$  is the fraction of encounters prey-predator where the prey dies.  $S_y$  is the area where each predator looks for the preys and  $S$  is the region where the preys are distributed. And  $p = \xi q$  where  $\xi$  is the quantity of new predators taken place by each consumed prey.

If we diminish the area, let us say that it exists  $0 < \phi < 1$  such that the not intervened area is  $\phi S < S$ , the consumption average changes to  $\frac{1}{\phi}q > q$ , then the rate of conversion of preys in new predators rate changes to  $\frac{1}{\phi}p > p$ .

### 3 Main results:

***The system when the area is modified from  $S$  to  $\phi S$  and anything is not made to maintain the carrying capacity of the means (You intervene and anything is not made to recover the carrying capacity of the means) it is expressed by:***

$$\begin{aligned} \frac{dX}{dt} &= r(1 - \frac{X}{K}) X - qXY - (\frac{1}{\phi} - 1)(qXY + \frac{rX^2}{K}) \\ \frac{dY}{dt} &= pXY - cY + (\frac{1}{\phi} - 1)pXY \end{aligned} \quad (2)$$

The equilibrium points are:  $(0, 0)$ ,  $(\phi K, 0)$  and  $(X_1^*, Y_1^*) = (\phi \frac{c}{p}, \phi \frac{r}{q}(\frac{pK-c}{pK})) = (\phi X_o^*, \phi Y_o^*)$

**Theorem 3.1.** *For the singularities of the system (2) one has that:*

- (a) *The singularity  $(0, 0)$  is saddle point*
- (b)  *$(\phi K, 0)$  saddle point, if and only if,  $K > \frac{c}{p}$ ; an attractor point, if and only if,  $K < \frac{c}{p}$  and an saddle-node attractor, if and only if,  $K \leq \frac{c}{p}$*
- (c) *If  $K > \frac{c}{p}$ , the singularity  $(X_1^*, Y_1^*) = (\phi \frac{c}{p}, \phi \frac{r}{q}(\frac{pK-c}{pK})) = (\phi X_o^*, \phi Y_o^*)$  is a locally asymptotically stable equilibrium point*

*Proof.* The Jacobian matrix of system (2) is

$$J(X; Y) = \begin{bmatrix} r(1 - \frac{2X}{\phi K}) - \frac{1}{\phi}qY & -\frac{1}{\phi}qX \\ \frac{1}{\phi}pY & \frac{1}{\phi}pX - c \end{bmatrix}$$

- (a) Evaluating the Jacobian matrix at  $(0, 0)$  we have that  $J(0, 0) = \begin{bmatrix} r & 0 \\ 0 & -c \end{bmatrix}$

As  $\text{Det}J(0, 0) = -rc < 0$ , then  $(0, 0)$  is saddle point.

- (b) The Jacobian of the system (2) evaluated at  $(\phi K, 0)$  is given by

$$J(\phi K, 0) = \begin{bmatrix} -r & -qK \\ 0 & pK - c \end{bmatrix},$$

the eigenvalues are:  $\lambda_1 = -r < 0$  and  $\lambda_2 = pK - c$ , then the sign of  $\lambda_2$  depends on the sign of  $pK - c$ . i.e.: hyperbolic saddle for  $K > \frac{c}{p}$  an attractor point, if and only if  $pK - c < 0$ .

(c) For the unique equilibrium point at the first quadrant we get:

$$J(X_1^*, Y_1^*) = \begin{bmatrix} -rc & -\frac{cq}{p} \\ \frac{r(pK-c)}{pK} & 0 \end{bmatrix}.$$

The  $\text{Trace}J(X_1^*, Y_1^*) = -rc < 0$  and  $\text{Det}J(X_1^*, Y_1^*) = \frac{rcq(pK-c)}{p^2K} > 0$ ,  
then  $(X_1^*, Y_1^*)$  an attractor point.

□

The equilibrium point to the interior of the first quadrant comes closer to the origin in the same proportion the intervened area. The conditions for the stabilities are similar to that of the original system, but with alterations it is easy to modify and to destroy both species.

**The system when it's modified the area from  $S$  to  $\phi S$  and stays the carrying capacity, it's expressed by:**

$$\begin{aligned} \frac{dX}{dt} &= r(1 - \frac{X}{K}) X - qXY - (\frac{1}{\phi} - 1)qXY \\ \frac{dY}{dt} &= pXY - cY + (\frac{1}{\phi} - 1)pXY \end{aligned} \quad (3)$$

where the equilibrium points are  $(0, 0)$ ,  $(K, 0)$  and

$$(X_2^*, Y_2^*) = (\phi \frac{c}{p}, \phi \frac{r}{q} (\frac{pK - \phi c}{pK})) = (\phi X_o^*, \phi Y_o^* + \phi \frac{r(1 - \phi)}{q})$$

**Theorem 3.2.** *The nature of the equilibrium points of the system (3). For all parameter values it has*

- (a) *The singularity  $(0, 0)$  is saddle point.*
- (b)  *$(K, 0)$  saddle point, if and only if,  $K > \frac{\phi c}{p}$ ; an attractor point, if and only if,  $K < \frac{\phi c}{p}$  and an saddle-node attractor, if and only if,  $K \leq \frac{\phi c}{p}$*
- (c) *If  $K > \frac{\phi c}{p}$ , the singularity  $(X_2^*, Y_2^*)$  is a locally asymptotically stable equilibrium point*

*Proof.* The Jacobian matrix of system (3) is

$$J(X; Y) = \begin{bmatrix} r(1 - \frac{2X}{K}) - \frac{1}{\phi}qY & -\frac{1}{\phi}qX \\ \frac{1}{\phi}pY & \frac{1}{\phi}pX - c \end{bmatrix}$$

- (a) Evaluating the Jacobian matrix at  $(0, 0)$  we have that  $\text{Det}J(0, 0) = -rc < 0$ , then  $(0, 0)$  is saddle point

- (b) Evaluating the Jacobian matrix at  $(K, 0)$  we have that the eigenvalues are:  $\lambda_1 = -r < 0$  and  $\lambda_2 = \frac{1}{\phi}pK - c$ , then the sign of  $\lambda_2$  depends on the sign of  $\frac{1}{\phi}pK - c$ . i.e.: saddle point for  $K > \frac{\phi c}{p}$  an attractor point, if and only if  $K < \frac{\phi c}{p}$
- (c) The Jacobian of the system (3) evaluated at  $(X_2^*, Y_2^*)$  has

$$\text{Trace}J(X_2^*, Y_2^*) = -\phi \frac{rc}{pK} < 0$$

and  $\text{Det}J(X_2^*, Y_2^*) = \phi \frac{rc(pK - \phi c)}{pK} > 0$ . Then  $(X_2^*, Y_2^*)$  an attractor point.

□

**The original system when the area has been modified from  $S$  to  $\phi S$  and you increases the carrying capacity to  $\frac{K}{\phi}$**  (You intervenes and it is made to recover the capacity of load of the means). The system of equations is expressed by:

$$\begin{aligned} \frac{dX}{dt} &= r(1 - \frac{X}{K})X - qXY - (1 - \phi)(\frac{rX^2}{K} - qXY) \\ \frac{dY}{dt} &= pXY - cY + (\frac{1-\phi}{\phi})pXY \end{aligned} \quad (4)$$

The equilibrium points:  $(0, 0)$ ,  $(\frac{1}{\phi}K, 0)$  and  $(X_3^*, Y_3^*) = (\phi \frac{c}{p}, \phi \frac{r}{q}(\frac{pK - \phi^2 c}{pK})) = (\phi X_o^*, \phi^3 Y_o^* + \phi \frac{r(1 - \phi^2)}{q})$

**Theorem 3.3.** *For the singularities of the system (4) one has:*

- (a) *The singularity  $(0, 0)$  is saddle point for all parameter values*
- (b) *The equilibrium point  $(\frac{1}{\phi}K, 0)$  is a saddle-node attractor (a nonhyperbolic equilibrium point) globally asymptotically stable if and only if  $K \geq \phi^2 \frac{c}{p}$ , a global attractor if and only if  $K < \phi^2 \frac{c}{p}$ ; in this case does not exist an equilibrium point at interior of the first quadrant.*
- (c) *If the unique positive equilibrium point  $(X_3^*, Y_3^*)$  is globally asymptotically stable if and only if,  $K > \phi^2 \frac{c}{p}$ ; in this case the equilibrium  $(\frac{1}{\phi}K, 0)$  is saddle point.*

*Proof.* (a) Evaluating the Jacobian matrix of system (4) at  $(0, 0)$  we have that  $\text{Det}J(0, 0) = -rc < 0$ , then  $(0, 0)$  is saddle point



- (b) Evaluating the Jacobian matrix of system (4) at  $(\frac{1}{\phi}K, 0)$  we have that the eigenvalues are:  $\lambda_1 = -r < 0$  and  $\lambda_2 = \frac{pK - \phi^2 c}{pK}$ , then the sign of  $\lambda_2$  depends on the sign of  $pK - \phi^2 c$ . i.e.: hyperbolic saddle for  $pK - \phi^2 c > 0$ , an attractor point for  $pK - \phi^2 c < 0$
- (c) For the unique equilibrium point at the first quadrant we get:

The  $Trace J(X_3^*, Y_3^*) = -\phi^2 \frac{rc}{pK} < 0$  and  $Det J(X_3^*, Y_3^*) = \frac{rc(pK - \phi^2 c)}{p^2 K^2} > 0$ . Then  $(X_3^*, Y_3^*)$  is attractor point.

□

When the area is modified from  $S$  to  $\phi S$  and the carrying capacity is increase to  $\frac{K}{\phi}$  with regard to the original system. The equilibrium point to the interior of the first quadrant comes closer to the origin. The preys diminish from  $\frac{c}{pK}$  to  $\phi \frac{c}{pK}$  and the predators diminishes from  $1 - \frac{c}{pK}$  to  $\phi(1 - \frac{\phi^2 c}{pK})$ . The stabilities stay but the conditions weaken.

## 4 Conclusions

It concludes by analyzing the dynamics of predator prey Lotka-Volterra, that modifying the  $S$  area at where they live species by construction activities of civil works, although they are taken or measures to mitigate negative impacts are implemented, it will always interfere the food chain, since in the equilibrium state of coexistence, the coordinate for the dams is the same as when nothing is done, but the condition is weaker when nothing is done, that when done to maintain capacity much less load and that increasing the capacity of the original. As  $X_3^* = X_2^* = X_1^* < X_o^*$  and  $Y_3^* \leq Y_2^* \leq Y_1^* < Y_o^*$  when modifying the area from  $S$  to  $\phi S$  and making any activity to maintain the carrying capacity of the means, the equilibrium is obtained when  $X^* = \frac{c}{ap} = \phi \frac{c}{p}$ , but the condition is weaker when we don't make anything that when we make to maintain the carrying capacity and a lot less than when increasing more the capacity of the original one. ( $\phi^2 \frac{c}{p} < \phi \frac{c}{p} < \frac{c}{p}$ ) If the original carrying capacity is low, it's necessary to make bigger effort for the survivals of the species.

It is convenient to take appropriate measures to improve the capacity of carry the ecosystem area.

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