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A Prey Predator Model with Vulnerable Infected Prey Consisting of Non-Linear Feedback

Vijaya Lakshmi Gandhavadi Mohan Rao [1], Suresh Rasappan [2], Regan Murugesan [3] and Vijaya Srinivasan [4]

[1] Department of Mathematics, Indira Institute of Engineering and Technology
Thiruvallur, Tamilnadu, India -631 203

[2, 3] Departments of Mathematics, Vel Tech University
Avadi, Tamilnadu, India-600 062

[4] Department of Mathematics, Annamalai University
Chidambaram, Tamilnadu, India-608 002

Abstract

This paper investigates the dynamical complexities of a prey predator model with susceptible and infected (SI) prey with nonlinear feedback. A nonlinear feedback mathematical model is proposed and analyzed to study the predator interaction with infected prey. It has been assumed that the susceptible and infected prey populations are predated by predator species in the presence of non linear feedback controls. By constructing a suitable Lyapunov function, global asymptotic stability is established. Finally, a numerical example is given and diagrams are presented which supports our results.

Keywords: Differential equations, Prey-predator model, Lyapunov function, SI model, Stability analysis

1. Introduction

The dynamical relationships between species and their complex properties are the heart of many ecological and biological processes. Predator-prey dynamics are well studied in the process of control of some ecosystems. In recent years, a significant number of the published papers on the mathematical discrete time models of biology discussed the system of differential equations and the associated numerical methods. Mathematical models on prey predator systems create a major interest during the last few decades. Study of such system with discrete models and continuous models can be found in [1-22].

Discrete time models give rise to more efficient computational models for numerical simulations and it exhibits more plentiful dynamical behaviours than a continuous time model of the same type. The pioneering work of Lotka and Volterra, the theoretical investigation of predator-prey systems in mathematical ecology has advanced greatly. An important extension was carried by Anderson and May, who divided the prey populations into susceptible and infected. One of the important conclusions that were obtained is that the existence of unhealthy prey can destabilized the balance between prey and predator.

The recent 50 years have seen an explosion of interest in the study of nonlinear dynamical systems. It is currently very much in vogue to study prey-predator system with functional responses and controls.

A functional response of the predator to the prey density in population dynamics refers to the change in the density of prey attached per unit time per predator as the prey density changes. Two species models like Holling type II, III and IV of predator to its prey have been extensively discussed in the literature. Leslie-Gower predator- prey model with variable delays, bifurcation analysis with time delay, global stability in a delayed diffusive system has been studied. Limit cycles for a generalized Gause type predator- prey model with functional response, three tropic level food chain system with Holling type IV functional responses, the discrete Nicholson Bailey model with Holling type II functional response and global dynamical behavior of prey-predator system has been revisited. Complex dynamic behavior of three species has been studies. Epidemiology models are routinely used nowadays to understand the spread of infectious diseases with the goal to determine vaccination policies possible to eradicate them, the SIR models, the effect of exotic species on a system of native prey-predator populations has been studied. [23-27].

The purpose of this paper is to study the dynamics of existence of infected prey effect the prey predator relationship with non-linear feedback control [28-32]. In this we assumed the susceptible and infected prey populations are predated by predator species by introducing modified Michaelis – Menten – Holling functional response [33].

Various Numerical simulations are then carried out to investigate the relationships between parameters of the model and the effects of the disease on the species.

This paper is organized as follows: In section 2 we have given problem statement. In section 3, we introduce the model and presented the related diagrams.

In section 4 we introduce the nonlinear feedback controls and obtain Global stability conditions by constructing Lyapunov function. In section 5, we presented some numerical simulations that carried out to investigate the relationships between parameters of the model and the effects of the disease on the populations. Finally, the last section 6, is devoted to the conclusion and remarks.

2. Problem Statement

Consider the chaotic system described by the dynamics

$$\dot{x} = Ax + f(x) + u \tag{1}$$

where $x \in R^n$ is the state of the system, A is the $n \times n$ matrix of the system parameters, the matrix A have some unknown parameters. $f : R^n \to R^n$ is the nonlinear part of the system. $u \in R^n$ is the adaptive feedback controller.

The global control problem is essentially to find feedback controller u, so as to stabilize the dynamics (1) for all initial conditions $x(0) \in R^n$, i.e. $\lim_{t\to\infty} \|x(t)\| = 0$ for all initial conditions $x(0) \in R^n$.

Lyapunov function methodology is used for establishing the feedback control of the system (1).

By the Lyapunov function methodology, a candidate Lyapunov function is taken as

$$V(x) = x^T Px \tag{2}$$

where P are $n \times n$ positive definite matrix.

Note that $V : R^n \to R^n$ is a positive definite function by construction. It is assumed that the parameters of the system (1) are measurable.

If a controller u found such that

$$\dot{V}(x) = -x^T Qx \tag{3}$$

where Q are positive definite matrix, then $\dot{V}$ is a negative definite function.

Hence, by Lyapunov stability theory (Hahn, 1967,[34]), the dynamics (1) is globally exponentially stable and hence the condition $\lim_{t\to\infty} \|x(t)\| = 0$ will be satisfied for all initial conditions $x(0) \in R^n$.

Then the states of the system (1) will be globally asymptotically stable.

3. System Description

The model can be written as

$$\begin{aligned} \dot{X}_1 &= rX_1(1-\frac{X_1}{K}) - PX_1X_2 \\ \dot{X}_2 &= PX_1X_2 - \frac{\gamma X_2 X_3}{X_3+\gamma\beta X_2} \\ \dot{X}_3 &= \frac{e\gamma X_2 X_3}{X_3+\gamma\beta X_2} - dX_3 \end{aligned} \tag{4}$$

here X_1 is the susceptible prey population, X_2 is the infected prey, X_3 is the predator, r is the growth rate, K the carrying capacity, P the incidence rate, γ the total attack rate for predator, β the handling time, e the conversion efficiency and d the death rate of predator. Also, $\frac{\gamma X_2 X_3}{X_3+\gamma\beta X_2}$ and $\frac{e\gamma X_2 X_3}{X_3+\gamma\beta X_2}$ are the Michaelis-Menten-Holling functional and numerical response. Let us we reduce the number of parameters by $x_1 = \frac{X_1}{K}$, $x_2 = \frac{Y_2}{K}$, $x_3 = \frac{X_3}{\gamma\beta K}$, $t = r\tau$ and we suppose $k = \frac{PK}{r}$, $b = \frac{\gamma}{r}$, $c = \frac{e}{r\beta}$, $a = \frac{d}{r}$

The modified Michaelis-Menten-Holling prey predator with vulnerable infected prey dynamics [33] is described by

$$\begin{aligned} \dot{x}_1 &= x_1(1-x_1) - kx_1x_2 \\ \dot{x}_2 &= kx_1x_2 - b\frac{x_2x_3}{x_3+x_2} \\ \dot{x}_3 &= c\frac{x_2x_3}{x_3+x_2} - ax_3 \end{aligned} \tag{5}$$

where $a = 0.2, b = 0.9, c = 0.2$ are constant parameters and k is varying parameter. When $0 < k < 0.6$, the system reached the equilibrium states. The equilibrium points are $(1,0,0)$.

Figure 1 depicts the equilibrium trajectories of the system [33].

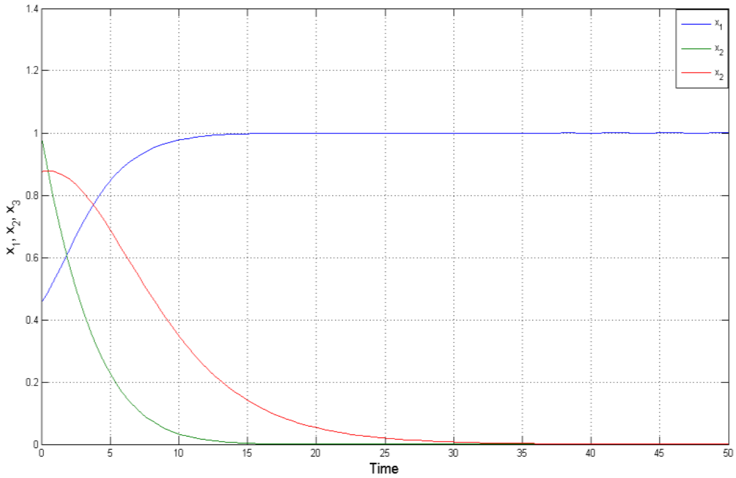

Figure 1 Equilibrium 0 < k < 0.6

When $0.6 < k < 1.8$, the system reached the equilibrium states (x^*, y^*, z^*). The equilibrium point depends on the value of k. Figure 2 and Figure 3 depict the equilibrium trajectories of the system at $k = 0.75$ [33].

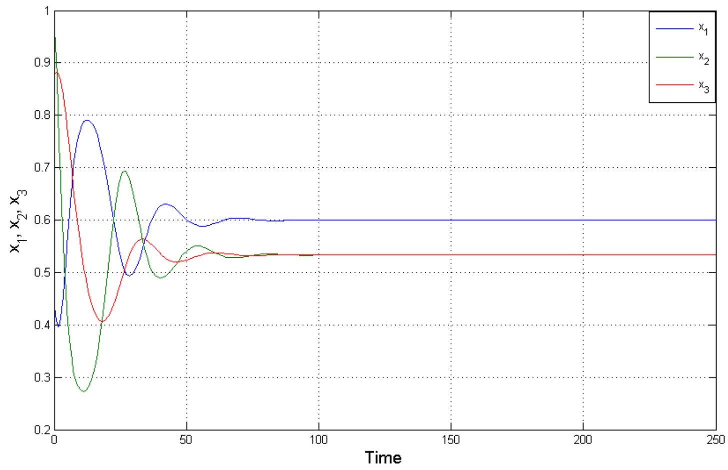

Figure 2 when 0.6 < k < 0.8, k= 0.75

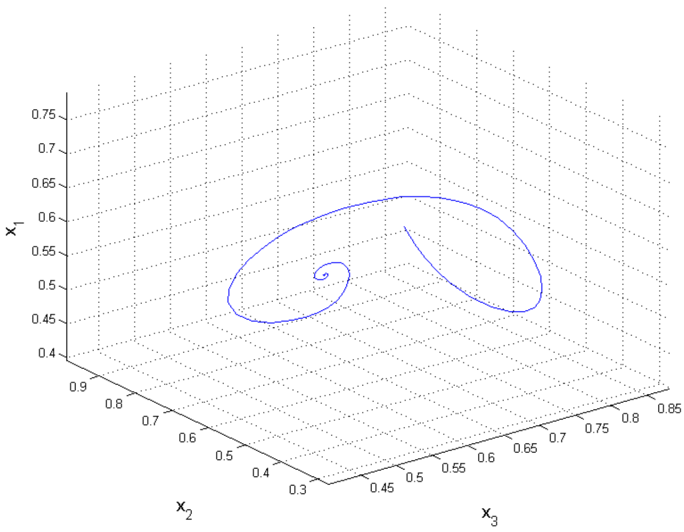

Figure 3 when 0.6 < k < 0.8, k= 0.75

When $1.8 < k < 1.9$, the system reached the dynamical behaviours such as holf bifurcation, limit cycles etc. Figure 4 depict the stable limit cycle of the system at $k = 1.85$ [33].

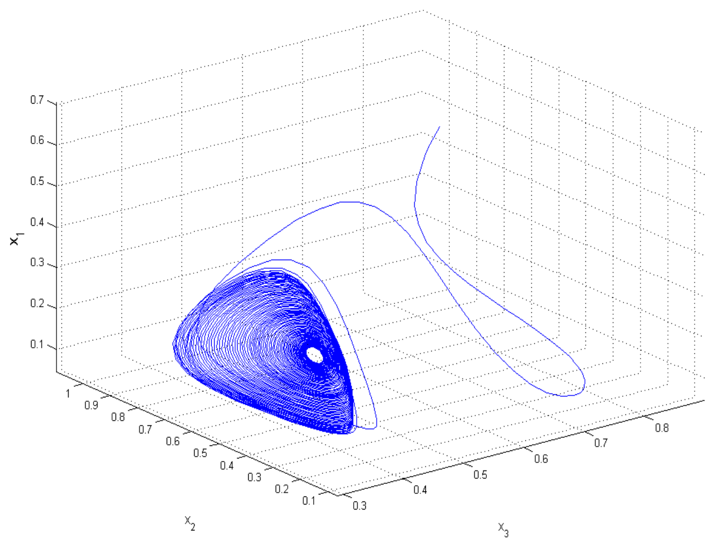

Figure 4 k = 1.85

4. A Prey Predator Model with Vulnerable Infected Prey consisting of Non-linear feedback controls

The modified Michaelis-Menten-Holling prey predator with vulnerable infected prey dynamics [33] is described by

$$\begin{aligned}\dot{x}_1 &= x_1(1-x_1) - kx_1x_2 + u_1\\ \dot{x}_2 &= kx_1x_2 - b\frac{x_2x_3}{x_2+x_3} + u_2\\ \dot{x}_3 &= c\frac{x_2x_3}{x_2+x_3} - ax_3 + u_3\end{aligned} \tag{6}$$

where x_1, x_2, x_3 are the state variables and a, b, c, k are positive parameters.

In this paper, we introduce the adaptive feedback procedure to design the controllers u_1, u_2, u_3. Where u_1, u_2, u_3 are feedback controllers, which is the function of the state variables. As long as these feedbacks stabilize system (6) converge to zero as the time t goes to infinity. That means that, this gives the system (6)
$\lim_{t\to\infty}\|x(t)\| = 0$

The candidate Lyapunov function is taken as

$$V(x_1, x_2, x_3) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2 \tag{7}$$

Differentiating (7) along the trajectories of the system (6), the simple calculation gives

$$\begin{aligned}\dot{V}(x_1, x_2, x_3) &= x_1(x_1(1-x_1) - kx_1x_2 + u_1)\\ &\quad + x_2(kx_1x_2 - b\frac{x_2x_3}{x_2+x_3} + u_2) + x_3(c\frac{x_2x_3}{x_2+x_3} - ax_3 + u_3)\end{aligned} \tag{8}$$

We defined the feedback controllers as follows

$$\begin{aligned}u_1 &= kx_1x_2 - 2x_1 + x_1^2\\ u_2 &= \frac{x_2x_3}{x_2+x_3} - x_2 - kx_1x_2\\ u_3 &= -c\frac{x_2x_3}{x_2+x_3}\end{aligned} \tag{9}$$

Substituting equation (9) into (8), then its implies that

$$\dot{V} = -x_1^2 - x_2^2 - a x_3^2 \tag{10}$$

which is a negative definite function.

Thus by Lyapunov stability theory [33], the modified Michaelis-Menten-Holling prey predator with vulnerable infected prey dynamics (6) is asymptotically stable.

Theorem 1: The modified Michaelis-Menten-Holling prey predator with vulnerable infected prey dynamics (6) is asymptotically stable with the feedback (9).

5. Numerical Simulation

In this section, we undertake the numerical simulations of the prey-predator system with vulnerable infected prey consisting of nonlinear feedback.

For the numerical simulations, the fourth order Runge-Kutta method is used to solve the differential equations (6) with the feedback controllers u_1, u_2, u_3.

The initial values of the master system (5) are chosen as

$$x_1(0) = 0.125, \quad x_2(0) = 0.625, \quad x_3(0) = 0.941$$

Figure 5 depicts the stable equilibrium points due to the nonlinear feedbacks u_1, u_2, u_3.

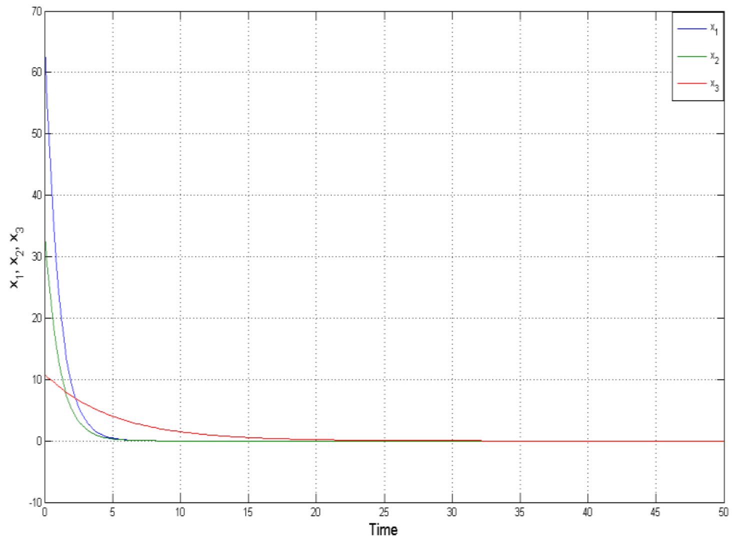

Figure 5 the stable equilibrium points due to the nonlinear feedbacks.

6. Conclusion

In this paper, we proposed a mathematical model to study the effect of susceptible, infected (SI) prey and the predator interactions by introducing Michaleis – Menten – Holling functional response as a dynamical system. We show that the three species prey-predator populations can be asymptotically stabilized using a nonlinear feedback control inputs, that is, the system (6) is examined via the techniques of global asymptotic stability analysis by constructing Lyapunov function. Numerical simulations were done to observe the effect of the disease on the three species and diagrams were presented which are supporting our results.

References

[1] Agiza. H. N, Elabbasy. E. M, EK-Metwally,Elsadany. A. A, *Chaotic dynamics of a discrete prey-predator model with Holling type II*, Nonlinear analysis: Real world applications 10 (2009), pp.116-129.
http://dx.doi.org/10.1016/j.nonrwa.2007.08.029

[2] Aziz-Alaoui. M.A and Daher Okiye. M, *Boundedness and global stability for a predator-prey model with modified Leslie-gower and Holling-type II schemes*, Applied Mathematics Letters 16 (2003), pp.1069-1075.
http://dx.doi.org/10.1016/s0893-9659(03)90096-6

[3] Baba I. Camara and Moulay A. Aziz-Alaoui, *Complexity in a prey predator model*, International conference in honor of Claude Lobry, Vol.9 (2008), pp.109-122.

[4] Debasis mukherjee, Prasenjit das and Dipak kesh, *Dynamics of a plant-herbivore model with Holling type II functional response.*

[5] Hongying lu and Weiguo wang, *Dynamics of a delayed discrete semi-ratio dependent predator-prey system with Holling type IV functional response*, Advances in difference equations (2011), pp.3-19.
http://dx.doi.org/10.1186/1687-1847-2011-7

[6] Lei zhang, Weiming wang, Yakui xue and Zhen jin, *Complex dynamics of a Holling-type IV predator-prey model,* (2008), pp.1-23.

[7] Manju Agarwal, Rachana Pathak, *Harvesting and hopf bifurcation in a prey-predator model with Holling type IV functional response*, International Journal of Mathematics and Soft Computing, Vol.2, No.1 (2012), pp.83-92.

[8] Moghadas. S.M, and Corbett. B.D, *Limit cycles in a generalized Gause-type predator-prey model,* Chaos, Solitons and Fractals 37 (2008), pp.1343-1355. http://dx.doi.org/10.1016/j.chaos.2006.10.017

[9] Shanshan chen, Junping shi and Junjie wei, *Global stability and hopf bifurcation in a delayed diffusive Leslie-gower predator-prey system,* International journal of bifurcation and chaos, Vol.22, No.3 (2012), pp.1-11. http://dx.doi.org/10.1142/s0218127412500617

[10] Shanshan chen, Junping shi, Junjie wei, *The effect of delay on a diffusive predator-prey system with Holling type-II predator functional response,* Communications on pure and applied analysis, Vol.12, No.1 (2013), pp.481-501. http://dx.doi.org/10.3934/cpaa.2013.12.481

[11] Shigui Ruan and Dongmei Xiao, *Global analysis in a predator-prey system with nonmonotonic functional response,* Society for industrial and applied mathematics, Vol.61, No.4 (2001), pp.1445-1472. http://dx.doi.org/10.1137/s0036139999361896

[12] Tianwei zhang, Xiaorong gan, *Existence and permanence of almost periodic solutions for Leslie-Gower predator-prey model with variable delays,* Electronic Journal of Differential Equations, No.105 (2013), pp. 1-21.

[13] Wen Zhang, Haihong Liu, Chenglin Xu, *Bifurcation analysis for a Leslie-gower predator-prey system with time delay,* International Journal of Nonlinear Science, Vol.15, No.1 (2013), pp.35-44.

[14] Yujing Gao, *Dynamics of a ratio-dependent predator-prey system with a strong Allee effect,* Discrete and Continuous Dynamical Systems series B, Vol.18, No.9 (2013), pp.2283-2313. http://dx.doi.org/10.3934/dcdsb.2013.18.2283

[15] Zhang Zi-zhen, Yang Hui-zhong, *Hopf bifurcation in a delayed predator-prey system with modified Leslie-gower and Holling type III schemes,* Acta Automatica Sinica, Vol.39, No.5 (2013), pp.610-616. http://dx.doi.org/10.3724/sp.j.1004.2013.00610

[16] Athul Johri, Neethu Trivedi, Anjali Sisodiya, Bijendra sing and Suman Jain, *Study of a prey-predator model with diseased prey,* Int. J. Contemp. Math. Sciences, Vol.7, No.10 (2012), pp.489-498.

[17] Sunitha Gakkhar, Ra`id Kamel Naji, *Chaos in three species ratio dependent food chain,* Chaos, Solitons and Fractals (2002) 14, pp.771-778. http://dx.doi.org/10.1016/s0960-0779(02)00038-3

[18] Sunitha Gakkhar, Brahampal Singh, *Complex dynamic behavior in a food*

web consisting of two preys and a predator, Chaos, Solitons and Fractals (2005) 24, pp.789-801. http://dx.doi.org/10.1016/j.chaos.2004.09.095

[19] Sunitha Gakkhar, Brahampal Singh, Raid Kamel Naji, *Dynamical behaviour of two predators competing over a single prey*, Biosystems (2007) 90, pp.808-817. http://dx.doi.org/10.1016/j.biosystems.2007.04.003

[20] Sunitha Gakkhar, Brahampal Singh, *The dynamics of a food web consisting of two prey and a harvesting predator*, Chaos, Solitons and Fractals (2007) 34, pp.1346-1356. http://dx.doi.org/10.1016/j.chaos.2006.04.067

[21] Shuwen zhang, Lansun chen, *A Holling II functional response food chain model with impulsive perturbation*, Chaos, Solitons and Fractals (2005) 24, pp. 1269-1278. http://dx.doi.org/10.1016/j.chaos.2004.09.051

[22] Shuwen zhang, Fengyan Wang, Lansun Chen, *A food chain model with impulsive perturbations and Holling IV functional response*, Chaos, Solitons and Fractals (2005) 26, pp. 855-866. http://dx.doi.org/10.1016/j.chaos.2005.01.053

[23] Abdelilah Kaddar, Abdelhadi Abta, Hamad Talibi Alaoui, *A comparisonof delayed SIF and SEIR epidemic models*, Nonlinear Analysis, Modelling and Control, Vol.16, No.2 (2011) pp.181-190.

[25] O.P. Misra, Pramod Kushwah, Chhatrapal Singh Sikarwar, *Effect of exotic species on a system of native prey-predator populations: A model*, American Journal of Computational and Applied Mathematics, 2(6) (2012), pp. 290-299. http://dx.doi.org/10.5923/j.ajcam.20120206.07

[26] Sudipa Chauhan, Om Prakash Misra, Joydip Dhar, *Stability analysis of SIR model with vaccination*, American Journal of Computational and Applied Mathematics, 4(1) (2014), pp.17-23.

[27] Pierre Auger, Rachid Mchich, Tanmay Chwodhury, Gauthier Sallet, Maurice Tchuente, Joydev Chattopadhyay, *Effects of disease affecting a predator on the dynamics of a predator-prey system*, Journal of Theoretical Biology, 258 (2009), pp.344-351. http://dx.doi.org/10.1016/j.jtbi.2008.10.030

[28] Sundarapandian, V., Suresh, R., (2010) "New results on the global chaos synchronization for Liu-Chen-Liu and Lu chaotic systems," PEIE 2010, CCIS Vol. 102, Springer-Verlag Berlin Heidelberg, 20–27. http://dx.doi.org/10.1007/978-3-642-15739-4_4

[29] Sundarapandian, V., Suresh, R., (2012) "Hybrid synchronization of Arneodo and Rossler chaotic systems by active nonlinear control," CCSIT 2012, Part-I, LNICST Vol. 84, Springer Heldelberg, Dordrecht, London, Newyork, 73–82.

http://dx.doi.org/10.1007/978-3-642-27299-8_8

[30] Sundarapandian V., and Suresh R, (2012) "Global Chaos Synchronization of Chen-Lee Systems via Backstepping Control," IEEE Digital Library, IEEE Catalog No: CFP 1277 R-CDR), 83-87.

[31] Suresh R. and Sundarapandian V., (2012) "Hybrid synchronization of n-scroll Chua and Lure chaotic system using backstepping control via novel feedback," Archives of Control Science, Vol. 22(LVIII), 255-278.

[32] Suresh, R. and Sundarapandian, V.,(2012) "Global Chaos synchronization of hyperchaotic Chen-Lee system using backstepping control via recursive feedback," Springer Verlag, Lecture Notes in Computer Science , CCIS 305, pp. 212–221, J. Mathew et al. (Eds.).

[33] S. A. Wuhaib, Y. Abu Hasan, A prey predator model with vulnerable infected prey, Applied Mathematical Sciences, Vol. 6, 2012, no. 107, 5333 – 5348.

[34] W. Hahn, The Stability of Motion, Berlin, Germany: Springer-Verlag, 1967.

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