Shadowing Property of Coupled Nonlinear Dynamical System

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Abstract

This paper aims to study the shadowing property of coupled nonlinear dynamical system obtained by combining the fast and slow versions of the three-parameter Lorenz system. Under the assumption that numerical solution has no errors, “pseudo-orbits” are generated by variations in the system’s parameters using the approach developed for computing sensitivity coefficients in chaotic systems.

Keywords: dynamical system, shadowing property, chaotic dynamics

1 Introduction

Dynamical systems theory provides a very powerful mathematical framework for understanding and exploring various natural and man-made complex systems whose behavior is changing over time [3]. The application areas of the dynamical systems theory are very diverse and multidisciplinary, and cover geosciences, biology, physics, chemistry, engineering, finance and other areas of human intellectual activities. One of the most rapidly developing components of the global theory of dynamical systems is the theory of pseudo-orbit shadowing in dynamical systems [5, 6]. Pseudo- (or approximate-) trajectories arise due to the presence of round-off errors, method errors, and other errors in computer simulation of dynamical systems. Consequently, in numerical modeling we can
compute a trajectory that is very close to an exact solution and the resulting approximate solution will be a pseudotrajectory. The shadowing property (or POTP, pseudo-orbit tracing property) means that near an approximate trajectory, there exists the exact trajectory of the system under consideration, such that it lies uniformly close to a pseudotrajectory.

The shadowing property of dynamical systems is a fundamental feature of hyperbolic systems [1, 2]. However, most physical systems are non-hyperbolic. Despite the fact that much of shadowing theory has been developed for hyperbolic systems, there is, however, evidence that non-hyperbolic attractors also have the shadowing property. In theory this property should be verified for each particular dynamical system, but this is more easily said than done.

In this paper, we consider the shadowing property of coupled nonlinear chaotic dynamical system obtained by combining the fast and slow versions of the three-parameter Lorenz system [4]. Under the assumption that numerical solution has no errors, “pseudo-orbits” are generated by variations in the system’s parameters using the approach that was developed for computing sensitivity coefficients in chaotic dynamical systems [8].

2 Essentials of the Shadowing

Let \((X,d)\) be a metric space, where \(X \subseteq \mathbb{R}^n\) and \(d\) is a distance function. Let us consider a discrete-time dynamical system, generated by iterating a map \(f : X \rightarrow X\), that is,

\[x_{m+1} = f(x_m), \quad m \in \mathbb{Z}_+.\]  

(1)

Given the system state \(x_0 \in X\) at time \(t = 0\), we define the trajectory of \(x_0\) under \(f\) to be the sequence of points \(\{x_m \in X : m \in \mathbb{Z}_+\}\) such that \(x_m = f^m(x_0)\), where \(f^m\) indicates the \(m\)-fold composition of \(f\) with itself, and \(f^0(x) \equiv x\). Thus, given the map \(f\) and the initial condition \(x_0\), equation (1) uniquely specifies the orbit of dynamical system. If \(x_m\) is the state of dynamical system at time \(t_m\), then the correct state at the next time \(t_{m+1}\) is given by \(f(x_m)\). In practice, however, there is a discrepancy between \(x_{m+1}\) and \(f(x_m)\) due to round-off errors and other truncation errors. Consequently, in computer simulations we can compute only a pseudo-orbit (or approximate trajectory) which can be shadowed by a real orbit.

A set of points \(\{x_m \in X : m \in \mathbb{Z}_+\}\) is a \(\delta\)-pseudotrajectory \((\delta > 0)\) of \(f\) if

\[d(x_{m+1}, f(x_m)) < \delta, \quad m \in \mathbb{Z}_+.\]  

(2)

We say that \(f\) satisfies shadowing property if given \(\varepsilon > 0\) there is \(\delta > 0\) such that for any \(\delta\)-pseudotrajectory \(\{x_m\}_{m=0}^{\infty}\) there exists a corresponding trajectory \(\{f^m(y)\}_{m=0}^{\infty}\), such that

\[d(x_m, f^m(y)) < \varepsilon, \quad m \in \mathbb{Z}_+.\]  

(3)

The shadowing property plays a crucial role in the exploration of the stability of
system dynamics. One of the most important theoretical results for hyperbolic dynamical systems is the shadowing lemma [1, 2], which states that for each nonzero distance $\varepsilon > 0$, there exists $\delta > 0$ such that each $\delta$-pseudotrajectory can be $\varepsilon$-shadowed. Informally, the shadowing lemma states that each pseudo-orbit stays uniformly close to a certain true trajectory.

The definition of pseudotrajectory and shadowing lemma for flows (continuous dynamical systems) are more complicated than for discrete dynamical systems [5, 6]. Let $f : X \rightarrow X$ be a vector field on $X$ of class $C^1$. Let $\phi : R \times X \rightarrow X$ be a flow on $X$ generated by $f$. A function $x : R \rightarrow X$ is called a $\delta$-pseudotrajectory if

$$d\left(\phi(t,x(\tau)),x(t+\tau)\right) < \varepsilon, \ 0 \leq t \leq 1, \ t, t+\tau \in R.$$  

The “continuous” shadowing lemma ensures that for the vector field $f$ generating the flow $\phi(t,x)$, the shadowing property holds in a small neighborhood of a compact hyperbolic set for dynamical system $\phi(t,x)$. However, the shadowing problem for continuous dynamical systems requires reparameterization of shadowing trajectories because for continuous dynamical systems close points of pseudo-orbit and true trajectory do not correspond to the same moments of time. A monotonically increasing homeomorphism $h : R \rightarrow R$ such that $h(0) = 0$ is called a reparameterization and denoted by $\text{Rep}$. For $\varepsilon > 0$, $\text{Rep}(\varepsilon)$ is defined as follows [6]:

$$\text{Rep}(\varepsilon) = \left\{ h \in \text{Rep} : \left| \frac{h(t_2) - h(t_1)}{t_2 - t_1} - 1 \right| \leq \varepsilon \right\}, \ \forall t_1, t_2 \in R.$$  

3 Calculation of pseudo-orbits generated by parameter variations

Let $x = f(x, \alpha)$ be an autonomous one-parameter dynamical system. Let us introduce the following transform [8]: $x'(x) = x + \delta x(x)$, where $x$ is a true orbit and $x'$ is a pseudo-orbit generated due to variation in the parameter $\alpha$. It can be shown, that $\delta f(x) = A\delta x(x)$, where

$$A = \left[ -\frac{\partial f}{\partial x} + \frac{d}{dt} \right]$$  

is a “shadow” operator. Thus, to find a pseudo-orbit we need to solve the equation $\delta x = A^{-1}\delta f$, i.e. we must numerically invert the operator $A$ for a given $\delta f$. To solve this problem, functions $\delta x$ and $\delta f$ are decomposed into their constituent Lyapunov covariant vectors $\nu_1(x), \ldots, \nu_n(x)$:

$$\delta x(x) = \sum_{i=1}^{n} \psi_i(x) \nu_i(x), \ \delta f(x) = \sum_{i=1}^{n} \phi_i(x) \nu_i(x).$$  

Each vector $\nu_i(x)$ satisfies the following equation:
\[
\frac{d\psi_i(x(t))}{dt} = \frac{\partial f}{\partial x} \psi_i(x(t)) - \lambda_i \psi_i(x(t)),
\]  
\tag{8}

where \( \lambda_1, \ldots, \lambda_n \) are the Lyapunov exponents. From the equation (6) we can obtain

\[
A(\psi, \nu) = \begin{bmatrix}
-\psi_i(x) \frac{\partial f}{\partial x} + \frac{d\psi_i(x)}{dt} \bigg|_{\phi_{i}} + \psi_i(x) \frac{d\psi_i(x)}{dt} \bigg|_{\phi_{i}}
\end{bmatrix} \nu_i(x).
\]  
\tag{9}

Substituting (8) into the last term of equation (9), we get

\[
A(\psi, \nu) = \begin{bmatrix}
\frac{d\psi_i(x)}{dt} - \lambda_i \psi_i(x)
\end{bmatrix} \nu_i(x).
\]  
\tag{10}

Combining equations (7), (10) and \( \delta f(x) = A\delta x(x) \), we can obtain

\[
\delta f(x) = \sum_{i=1}^{n} A(\psi, \nu) = \sum_{i=1}^{n} \left( \frac{d\psi_i(x)}{dt} - \lambda_i \psi_i(x) \right) \nu_i.
\]  
\tag{11}

This equation gives the following relationship between \( \psi_i(x) \) and \( \phi_i \) along the orbit:

\[
\frac{d\psi_i(x)}{dt} = \phi_i(x) + \lambda_i \psi_i(x).
\]  
\tag{12}

Thus, we can calculate \( \psi_i(x) \) using the equation (12) by first decomposing \( \delta f \) as a sum (7), and then the desired \( \delta x \) can be obtained from the equation (7). If dynamical system has a zero Lyapunov exponent, \( \lambda_0 = 0 \), then we need to introduce a time-dilation variable \( \mu \) that satisfies the following equation [8]:

\[
\mu + \psi_0(x) = 0, \quad \psi_0(x) = \lim_{T \to \infty} \frac{1}{T} \left[ \psi_0(x(T)) - \psi_0(x(0)) \right].
\]  
\tag{13}

In the presence of the variable \( \mu \), the expression for calculating \( \delta x \) takes the following form: \( \delta x = A^{-1} (\delta f + \mu f) \). The supplement \( \mu f \) affects the equation (12) only for \( \lambda_0 = 0 \):

\[
\frac{d\psi_i^0(x)}{dt} = \phi_i^0(x) + \mu.
\]

Thus, the procedure for calculating a pseudo-orbit represents the following set of steps:

1. Obtain initial conditions of the system attractor by integrating the model equations from \( t_0 = -20 \) to \( t_0 = 0 \), beginning from random initial conditions.
2. Solve the model equation to obtain a trajectory \( x(t) \), \( t \in [0, 20] \) on the attractor.
(3) Compute the Lyapunov exponents $\lambda_i$ and the Lyapunov covariant vectors $v_i(x(t))$, $i=1,...,n$.

(4) Define $\delta f = (\partial f / \partial \alpha) \delta \alpha$, where the variation in the parameter $\alpha$ is taken to be $\delta \alpha = 0.1 \times \alpha$, and execute the Lyapunov spectrum decomposition of $\delta f$ along the trajectory $x(t)$ to obtain $\varphi_i(x)$, $i=1,...,n$.

(5) Calculate the time dilation variable $\mu$ using equation (13).

(6) Compute $\delta x$ along the trajectory $x(t)$.

(7) Calculate the pseudotrajectory as $x'(x) = x + \delta x(x)$.

4 Some results of numerical experiments

Coupled nonlinear dynamical system used in this paper, is obtained by combining the fast and slow versions of the original Lorenz model [4] and in operator form can be written as follows:

$$\dot{x} = (L + Q)x,$$

(14)

where the nonlinear uncoupled operator $L$ and linear coupled operator $Q$ are represented by the following matrices

$$L = \begin{bmatrix}
-\sigma & \sigma & 0 \\
-1 & -r & -x \\
0 & x & -b \\
0 & \varepsilon r & -\varepsilon \\
0 & \varepsilon X & -\varepsilon b
\end{bmatrix}, \quad Q = \begin{bmatrix}
-c & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & -c & 0
\end{bmatrix}.$$

The system’s state vector is $x = (x, y, z, X, Y, Z)^T$, where lower case letters represent the fast subsystem and capital letters – the slow subsystem; the parameter vector is $\alpha = (\sigma, r, b, c, \varepsilon)^T$, where $\sigma$, $r$, $b$ are the parameters of L63 model, $c$ is a coupling strength parameter and $\varepsilon$ is the time-scale factor. The parameter values are taken as $\sigma = 10$, $r = 28$, $b = 8/3$, $\varepsilon = 0.1$, $c \in [0.1; 1.2]$. Chosen values of $\sigma$, $r$ and $b$ correspond to chaotic dynamics of the L63 system and the parameter $\varepsilon = 0.1$ indicates that the slow system is 10 times slower than the fast system. System’s basic dynamical, correlative and spectral properties as well as numerical algorithm were considered in details in [7].
Fig 1 Original orbit (in red) and pseudo-orbit (in blue) for fast $z$ and slow $Z$ variables for $c=0.01$.

Fig 2 Differences between variables that correspond to the original trajectory and pseudo-orbit for $c=0.01$.

Fig 3 Original orbit (in red) and pseudo-orbit (in blue) for fast $z$ and slow $Z$ variables for $c=0.8$. 
Fig 4 Differences between variables that correspond to the original trajectory and pseudo-orbit for $c=0.8$.

We consider two sets of numerical experiments: weak coupling ($c=0.01$) and strong coupling ($c=0.8$) between fast and slow systems. Fast $z$ and slow $Z$ variables that correspond to the original and pseudo-orbits are shown in Fig. 1 when the coupling strength parameter $c=0.01$. The differences between state variables corresponding to the original orbits and pseudotrajectories of the fast and slow systems are plotted in Fig. 2. These figures show that the calculated pseudo-orbits are close to corresponding true trajectories over a specified time interval, demonstrating the shadowability. The strong coupling does not introduce significant qualitative and quantitative changes in the behavior of pseudo-orbits with respect to the true trajectories. The original and pseudo fast $z$ and slow $Z$ variables for $c=0.8$ are shown in Fig. 3, and the differences between fast and slow state variables are presented in Fig. 4.

5 Concluding remarks

In this paper, we considered the shadowing property of coupled nonlinear dynamical system obtained by combining the fast and slow versions of the three-parameter Lorenz system. “True” trajectories were calculated by ignoring numerical approximation errors (i.e. the system is assumed to be “ideal”). To compute “pseudo-orbits”, equations of the dynamical system were numerically integrated with infinitesimal perturbations in the parameters. Results obtained show that the dynamical system under consideration possesses the pseudo-orbit tracing property for chosen parameter values. The considered approach is important for analysing system sensitivity with respect to its parameters and for exploring the validity of solutions obtained by numerical integration of differential equations used to simulate various dynamical processes.

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