A Remark about Quasi 2-Normed Space

Aleksa Malčeski
Faculty of Mechanical Engineering
Ss. Cyril and Methodius University
Skopje, Macedonia

Risto Malčeski
Faculty of informatics
FON University
Bul. Vojvodina bb, Skopje, Macedonia

Katerina Anevksa
Faculty of informatics
FON University
Bul. Vojvodina bb, Skopje, Macedonia

Samoil Malčeski
Centre for research and development of education
Skopje, Macedonia

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Abstract

C. Park [6] did the generalization of the term of quasi-normed space, i.e. he introduced the term of a quasi 2-normed space. Moreover, C. Park proved few properties of quasi 2-norm, and M. Kir and M. Acikgoz [2] elaborated the procedure for completing the quasi 2-normed space. In this paper will be proven that each quasi 2-norm generates a family of quasi-norms and will be considered some properties of such derived quasi-normed spaces.

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1 Introduction

In 1965, S. Gähler introduced $2$-normed spaces. One of the axioms of the $2$-norm is the parallelepiped inequality, which is actually a fundamental one in the theory of $2$-normed spaces. C. Park replaced precisely this inequality (analogously as in the normed spaces) with a new condition, which actually means that he gave the following definition for quasi $2$-normed space.

Definition 1 ([6]). Let $L$ be a real vector space with $\dim L \geq 2$. Quasi $2$-norm is a real function $\| \cdot \| : L \times L \to [0, \infty)$ such that:

a) $\| x, y \| \geq 0$, for all $x, y \in L$ and $\| x, y \| = 0$ if and only if the set $\{x, y\}$ is linearly dependent;
b) $\| x, y \| = \| y, x \|$, for all $x, y \in L$;
c) $\| \alpha x, y \| = |\alpha| \cdot \| x, y \|$, for all $x, y \in L$ and for each $\alpha \in \mathbb{R}$, and
d) It exists a constant $K \geq 1$ such that $\| x + y, z \| \leq K(\| x, z \| + \| y, z \|)$, for all $x, y, z \in L$.

If $\| \cdot \|$ is a quasi $2$-norm, then the ordered pair $(L, \| \cdot \|)$ is called a quasi $2$-normed space. The smallest possible number $K$ such that it satisfies the condition $d$ is called a modulus of concavity of the quasi $2$-norm $\| \cdot \|$.

Further, M. Kir and M. Acikgoz [2] gave few examples of trivial quasi $2$-normed spaces and consider the question about completing the quasi $2$-normed space, and C. Park [6] characterized a quasi $2$-normed space, i.e. he proved the following theorem.

Theorem 1 ([6]). Let $(L, \| \cdot \|)$ be a quasi $2$-normed space. It exists $p$, $0 < p \leq 1$ and an equivalent quasi $2$-norm $\| \cdot \|$ over $L$ such that

$$\| x + y, z \| \leq \| x, z \| ^p + \| y, z \| ^p,$$

for all $x, y, z \in L$. ■

Definition 2 ([6]). Quasi $2$-norm defined as in Theorem 1 is called $(2, p)$–norm, and quasi $2$-normed space $L$ is called $(2, p)$–normed space.

2 Quasi-norms generated by quasi $2$-norm

Theorem 2. Let $(L, \| \cdot \|)$ be quasi $2$-normed space with constant $K \geq 1$, $p \geq 1$ and $\{a, b\}$ be linearly independent subset of $L$. Then
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\[ \| x \| = (\| x, a \|^p + \| x, b \|^p)^{1/p}, \quad x \in L \tag{2} \]
defines a quasi-norm over \( L \).

**Proof.** Clearly, \( \| x \| \geq 0 \) and \( \| 0 \| = 0 \). Let \( \| x \| = 0 \). Then since (2) it is true that \( \| x, a \| = \| x, b \| = 0 \), thus Definition 1 implies that the sets \( \{x, a\} \) and \( \{x, b\} \) are linearly dependent. Since the set \( \{a, b\} \) is linearly independent, it holds that \( tx = \alpha a \) and \( qx = \beta b \), for some \( t, q \neq 0 \). Thus, \( \alpha a = \beta b \) and since \( \{a, b\} \) is linearly independent and \( t, q \neq 0 \), the last equality implies \( \alpha = \beta = 0 \), i.e. \( x = 0 \).

Let \( x \in L \) and \( \alpha \in \mathbb{R} \). Since (2) we get that

\[ \| \alpha x \| = (\| \alpha x, a \|^p + \| \alpha x, b \|^p)^{1/p} = |\alpha| (\| x, a \|^p + \| x, b \|^p)^{1/p} = |\alpha| \cdot \| x \|. \]

Finally, the condition \( d) \) in Definition 1 and Minkowski inequality imply that

\[
\| x + y \| = (\| x, a \|^p + \| y, a \|^p + \| x, b \|^p + \| y, b \|^p)^{1/p} \\
\leq \{K(\| x, a \| + \| y, a \|)^p + K(\| x, b \| + \| y, b \|)^p\}^{1/p} \\
\leq K(\| x, a \|^p + \| x, b \|^p)^{1/p} + K(\| y, a \|^p + \| y, b \|^p)^{1/p} \\
= K(\| x \| + \| y \|),
\]
holds for all \( x, y \in L \). The latter actually means that (2) defines quasi-norm over \( L \), which in our further regarding will be denoted by \( \| \cdot \|_{a,b,p} \).

**Theorem 3.** Let \((L, \| \cdot \|)\) be a quasi 2-normed space with constant \( K \geq 1 \) and \( \{a, b\} \) be linearly independent subset of \( L \). Then

\[ \| x \| = \max\{\| x, a \|, \| x, b \|\}, \quad x \in L \tag{3} \]
defines a quasi-norm over \( L \).

**Proof.** It is obvious that \( \| x \| \geq 0 \) and \( \| 0 \| = 0 \). Let \( \| x \| = 0 \). Then (3) implies that \( \| x, a \| = \| x, b \| = 0 \), and analogously as the proof of Theorem 2 we get that \( x = 0 \). Let \( x \in L \) and \( \alpha \in \mathbb{R} \). Thereby (3) holds, it is true that

\[ \| \alpha x \| = \max\{\| \alpha x, a \|, \| \alpha x, b \|\} = \max\{|\alpha| \cdot \| x, a \|, |\alpha| \cdot \| x, b \|\} = |\alpha| \cdot \| x \|. \]

Further, the properties of maximum and the condition \( d) \) in Definition 1 imply the following inequalities
\[ \| x + y \| = \max \{ \| x + a \|, \| x + y \|, \| y + b \| \} \]
\[ \leq \max \{ K(\| x, a \| + \| y, a \|), K(\| x, b \| + \| y, b \|) \} \]
\[ \leq K(\max \{ \| x, a \|, \| x, b \| \} + \max \{ \| y, a \|, \| y, b \| \}) \]
\[ = K(\| x \| + \| y \|) . \]

The latter means that (3) defines a quasi-norm over \( L \). The above quasi-norm in our further considerations will be denoted by \( \| \cdot \|_{a,b,\infty} . \)

**Theorem 4.** Let \( (L, \| \cdot \|) \) be a quasi 2-normed space and let \( \{a, b\} \) be linearly independent subset of \( L \). Then for all \( p, q \geq 1 \) the quasi-norms \( \| \cdot \|_{a,b,p} , \| \cdot \|_{a,b,q} \) and \( \| \cdot \|_{a,b,\infty} \) are equivalent.

**Proof.** The proof of the theorem is analogous to the Proof of Theorem 3, [3]. \( \Box \)

**Problem 1.** Let \( \{a, b\} \) and \( \{c, d\} \) be two linearly independent sets in a quasi 2-normed space \( (L, \| \cdot \|) \) and let consider the families of quasi-norms \( \{\| \cdot \|_{a,b,\infty}, \| \cdot \|_{a,b,p}, p \geq 1 \} \) and \( \{\| \cdot \|_{c,d,\infty}, \| \cdot \|_{c,d,p}, p \geq 1 \} \). Are the quasi-norms equivalent and furthermore which conditions must be satisfied, the quasi-norms to be equivalent: \( \| \cdot \|_{a,b,\infty} \) and \( \| \cdot \|_{c,d,\infty} ; \) \( \| \cdot \|_{a,b,\infty} \) and \( \| \cdot \|_{c,d,p} , \ p \geq 1 ; \) and \( \| \cdot \|_{a,b,p} \) and \( \| \cdot \|_{c,d,q} , \ p, q \geq 1 \).

**Definition 3.** The sequence \( \{x_n\}_{n=1}^{\infty} \) at quasi 2-normed space \( (L, \| \cdot \|) \) is called convergent sequence if it exists \( x \in L \) so that \( \lim_{n \to \infty} \| x_n - x, y \| = 0 \), for each \( y \in L \). The vector \( x \in L \) is called a bound of the sequence \( \{x_n\}_{n=1}^{\infty} \) and is denoted as \( \lim_{n \to \infty} x_n = x \) or \( x_n \to x , \ n \to \infty \).

The sequence \( \{x_n\}_{n=1}^{\infty} \) at a quasi 2-normed space \( (L, \| \cdot \|) \) so that \( \lim_{m,n \to \infty} \| x_n - x_m, y \| = 0 \), for each \( y \in L \), is called Cauchy sequence.

A quasi 2-normed space \( (L, \| \cdot \|) \) in which each Cauchy sequence is convergent sequence is called quasi 2-complet (quasi 2-Banach) space.

**Theorem 5.** Let \( \{x_n\}_{n=1}^{\infty} \) be a sequence in quasi 2-normed space \( (L, \| \cdot \|) \) and let \( \{a, b\} \) be linearly independent set at \( L \).

a) If the sequence \( \{x_n\}_{n=1}^{\infty} \) is Cauchy sequence in \( (L, \| \cdot \|) \), then latter is also Cauchy in \( (L, \| \cdot \|_{a,b,p}) , \ p \geq 1 \) and \( (L, \| \cdot \|_{a,b,\infty}) \).

b) If the sequence \( \{x_n\}_{n=1}^{\infty} \) is convergent sequence in \( (L, \| \cdot \|) \), then latter is also convergent in \( (L, \| \cdot \|_{a,b,p}), \ p \geq 1 \) and \( (L, \| \cdot \|_{a,b,\infty}) \).
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Proof. The proof of the theorem is analogous to the Proof of Theorem 6, [3]. ■

Problem 2. Let \((L, \| \cdot \|)\) be a quasi 2-Banach space and \(\{a, b\}\) be linearly independent set in \(L\). Are the quasi-normed spaces \((L, \| \cdot \|_{a, b, p})\), \(p \geq 1\) and \((L, \| \cdot \|_{a, b, \infty})\) also quasi-Banach spaces and furthermore which conditions must be satisfied the above statement to holds true.

3 Inequalities in a quasi 2-normed space

Theorem 6. Let \((L, \| \cdot \|)\) be a quasi 2-normed space and \(K \geq 1\). Then
\[
\begin{align*}
\| x + y, z \| + \| x - y, z \| - \min\{\| x + y, z \|, \| x - y, z \|\} &\leq K(\| x, z \| + \| y, z \|) \quad \text{(4)} \\
\| x, z \| + \| y, z \| + \| x - y, z \| &\leq K(\| x + y, z \| + \| x - y, z \|) \quad \text{(5)}
\end{align*}
\]
are satisfied for all \(x, y, z \in L\).

Proof. Since the condition \(d)\) in Definition 1, the following inequalities hold true
\[
\begin{align*}
\| x + y, z \| + \| x - y, z \| - K(\| x, z \| + \| y, z \|) &\leq \| x - y, z \| \\
\| x + y, z \| + \| x - y, z \| - K(\| x, z \| + \| y, z \|) &\leq \| x + y, z \|
\end{align*}
\]
Therefore it is true that
\[
\| x + y, z \| + \| x - y, z \| - K(\| x, z \| + \| y, z \|) \leq \min\{\| x - y, z \|, \| x + y, z \|\}.
\]
The last one is equivalent to the inequality (4). Further, we consider the condition \(d)\) in definition 1, once again. Thereby the declared condition, it is true that
\[
\begin{align*}
2 \| x, z \| - \| x + y, z \| - \| x - y, z \| &\leq K(\| x + y, z \| + \| x - y, z \|) \\
2 \| y, z \| - \| x + y - (x - y), z \| &\leq K(\| x + y, z \| + \| x - y, z \|)
\end{align*}
\]
thus
\[
2 \max\{\| x, z \|, \| y, z \|\} \leq K(\| x + y, z \| + \| x - y, z \|). \quad \text{(6)}
\]
On the other hand
\[
\| x, z \| + \| y, z \| + \| x - y, z \| = 2 \max\{\| x, z \|, \| y, z \|\}. \quad \text{(7)}
\]
Finally, the equality (7) and the inequality (6) implicate inequality in (5). ■

Paper [4] consists of many inequality which are implicated by the parallelepiped inequality. One of them is the following statement
Theorem 7 ([4]). Let \((L, \| \cdot , \|)\) be 2-normed space. The inequalities
\[
\| x + y, z \| + (2 - \| x \| + \| y \|, z \|) \| y, z \| \leq \| x, z \| + \| y, z \|, \text{ and}
\| x, z \| + \| y, z \| \leq \| x + y, z \| + (2 - \| x \| + \| y \|, z \|) \| x, z \|.
\]
hold true, for all \(x, y, z \in L\) such that \(\| x, z \| \geq \| y, z \| > 0\).

The following Theorem proves two inequalities in quasi 2-normed space, which are analogies to the inequalities stated in Theorem 7.

Theorem 8. Let \((L, \| \cdot , \|)\) be a quasi 2-normed space and \(K \geq 1\). The inequalities
\[
\| x + y, z \| + K(2 - \| x \|/\| y, z \| + y, z \|) \| y, z \| \leq K(\| x, z \| + \| y, z \|), \quad (8)
\]
\[
K(\| x, z \| + \| y, z \|) \leq 2 + K(2K^2 - \| x \|/\| y, z \| + y, z \|) \| x, z \|. \quad (9)
\]
hold true, for all \(x, y, z \in L\) such that \(\| x, z \| \geq \| y, z \| > 0\).

Proof. Let \(\| x, z \| \geq \| y, z \|\). So,
\[
\| x + y, z \| = (\| x \| + \| y \|, z \|) \| y, z \| + (\| x \| \| y, z \| \| x, z \| \| y, z \|, z \|)
\leq K(\| x \| + \| y \|, z \| + \| x, z \| \| y, z \|, z \|)
= K(\| y, z \| \| x \| + \| y \|, z \| + K(\| x \| + \| y, z \| + \| x, z \|)
= K(y, z \| - 2 \| y, z \|)
= K(\| y, z \| - 2 \| y, z \| + K(\| x \| + \| y, z \|).
\]
Thus, the inequality (8) is true.

Since
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\[ \left\| \left( \frac{x}{\|x,z\|} + \frac{y}{\|y,z\|} \right) \right\| x, z, \| = \| x + \frac{y}{\|y,z\|} \right\| x, z, \| \]
\[ = \| x + y + \frac{y}{\|y,z\|} \right\| x, z, \| - \frac{y}{\|y,z\|} \right\| y, z, \| \]
\[ \leq K \| x + y, z \| + K \| \frac{y}{\|y,z\|} \right\| x, z, \| - \frac{y}{\|y,z\|} \right\| y, z, \| \]
\[ = K \| x + y, z \| + K (\| x, z \| + \| y, z \|) \]
\[ = K \| x + y, z \| - K (\| x, z \| + \| y, z \|) + 2K \| x, z \| \]

it is true that

\[ K (\| x, z \| + \| y, z \|) \leq K \| x + y, z \| + (2K - \| x, z \| - \| y, z \|) \leq K \| x, z \| \]
\[ = K (x - y, z) \leq K (x, z) \leq K (x, z) + (2K - \| x, z \|) \leq K (x, z) \]
\[ = K (x, z) \leq K (x, z) + (2K - \| x, z \|) \leq K (x, z) \]
\[ = K (x, z) \leq K (x, z) + (2K - \| x, z \|) \leq K (x, z) \]
\[ i.e. \text{ the inequality (9) is true. □} \]

**Theorem 9.** Let \((L, \| \cdot \|)\) be a quasi \((2, p)\)-normed space. For all \(z, x_1, x_2, \ldots, x_n \in L\) such that satisfy that \(\| x_1, z \| \geq \| x_2, z \| \geq \ldots \geq \| x_n, z \| > 0\) the following inequalities hold true

\[ \| x_1, z \|^p \sum_{i=1}^{n} \frac{x_i}{\|x_i, z\|^p} \| z \|^p - \sum_{i=1}^{n} (\| x_1, z \| - \| x_i, z \|)^p \leq \sum_{i=1}^{n} \frac{x_i}{\|x_i, z\|^p} \| z \|^p, \quad (10) \]

\[ \sum_{i=1}^{n} \| x_i, z \|^p \leq \| x_n, z \|^p \sum_{i=1}^{n} \frac{x_i}{\|x_i, z\|^p} \| z \|^p + \sum_{i=1}^{n} (\| x_i, z \| - \| x_n, z \|)^p, \quad (11) \]

for \(0 < p \leq 1\).

**Proof.** The inequality (1) directly implies that
\[ \| \sum_{i=1}^{n} \left( \frac{x_i}{\|x_i,z\|} \right) z \|^{p} = \left( \sum_{i=1}^{n} \frac{x_i}{\|x_i,z\|} + \frac{x_i}{\|x_i,z\|} \right) \left( \sum_{i=1}^{n} \frac{x_i}{\|x_i,z\|} - \sum_{i=1}^{n} \frac{x_i}{\|x_i,z\|} \right) z \|^{p} \]
\[ \leq \left( \sum_{i=1}^{n} \frac{x_i}{\|x_i,z\|} z \|^{p} + \sum_{i=1}^{n} \frac{x_i}{\|x_i,z\|} - \sum_{i=1}^{n} \frac{x_i}{\|x_i,z\|} \right) z \|^{p} \]
\[ = \frac{1}{\|x_1,z\|^{p}} \left( \sum_{i=1}^{n} x_i, z \|^{p} + \frac{1}{\|x_1,z\|^{p}} \sum_{i=1}^{n} \|x_1,z\| - \frac{1}{\|x_1,z\|^{p}} \sum_{i=1}^{n} \|x_1,z\| \right) z \|^{p} \]
\[ \leq \frac{1}{\|x_1,z\|^{p}} \left( \sum_{i=1}^{n} x_i, z \|^{p} + \frac{1}{\|x_1,z\|^{p}} \sum_{i=1}^{n} \|x_1,z\| - \frac{1}{\|x_1,z\|^{p}} \sum_{i=1}^{n} \|x_1,z\| \right) z \|^{p}. \]

i.e. the inequality (10) holds true. Further,

\[ \| \sum_{i=1}^{n} \frac{x_i}{\|x_i,z\|} z \|^{p} = \left( \sum_{i=1}^{n} \frac{x_i}{\|x_i,z\|} + \frac{x_i}{\|x_i,z\|} \right) \left( \sum_{i=1}^{n} \frac{x_i}{\|x_i,z\|} - \sum_{i=1}^{n} \frac{x_i}{\|x_i,z\|} \right) z \|^{p} \]
\[ \leq \left( \sum_{i=1}^{n} \frac{x_i}{\|x_i,z\|} z \|^{p} + \sum_{i=1}^{n} \frac{x_i}{\|x_i,z\|} - \sum_{i=1}^{n} \frac{x_i}{\|x_i,z\|} \right) z \|^{p} \]
\[ = \left( \sum_{i=1}^{n} \frac{x_i}{\|x_i,z\|} z \|^{p} + \frac{1}{\|x_1,z\|^{p}} \sum_{i=1}^{n} \|x_1,z\| \right) z \|^{p} \]
\[ \leq \left( \sum_{i=1}^{n} \frac{x_i}{\|x_i,z\|} z \|^{p} + \frac{1}{\|x_1,z\|^{p}} \sum_{i=1}^{n} \|x_1,z\| \right) z \|^{p}, \]

i.e. the inequality (11) holds true.

**Corollary 1.** Let \((L, \|\cdot, \|)\) be a quasi 2-normed space so that the quasi 2-norm satisfies the inequality (9). For all \(x, y, z \in L\) so that \(\|x, z\| \geq \|y, z\| > 0\), the following hold true

\[ \| x, z \|^{p} \leq \| y, z \|^{p} \left( \frac{x}{\|x,z\|} + \frac{y}{\|y,z\|} \right) + \| x, z \| - \| y, z \|^{p} \leq \| x, y, z \|^{p}, \]
\[ \| x + y, z \|^{p} \leq \| y, z \|^{p} \left( \frac{x}{\|x,z\|} + \frac{y}{\|y,z\|} \right) - \| x, z \| - \| y, z \|^{p}, \]

for \(0 < p \leq 1\).

**Proof.** Let \(n = 2, \ x_1 = x, \ x_2 = y\). Thus, the proof is directly implicated by Theorem 9.

In our further considerations we will generalize the inequalities (8) and (9) exposed (given, stated) in Theorem 8. In purpose of that we will firstly prove the validity of the following Lemma.

**Lemma 1.** Let \(L\) be a quasi 2-normed space with modulus of concavity \(K \geq 1\), thus for each \(n \geq 1\) and for all \(z, x_1, x_2, \ldots, x_n \in L\).
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\[ \| \sum_{i=1}^{n} x_i, z \| \leq K^{1+\lfloor \log_2(n-1) \rfloor} \left( \| x_1, z \| + \| x_2, z \| \right) \] (12)

holds true.

**Proof.** The statement will be proven by applying the principle of mathematical induction. Since the properties of a quasi 2-norm, it is true that

\[ \| x_1 + x_2, z \| \leq K \left( \| x_1, z \| + \| x_2, z \| \right) \]

\[ = K^{1+\lfloor \log_2(2-1) \rfloor} \left( \| x_1, z \| + \| x_2, z \| \right) \]

\[ \| x_1 + x_2 + x_3, z \| \leq K \left( \| x_1, z \| + \| x_2 + x_3, z \| \right) \]

\[ \leq K \| x_1, z \| + C^2 \left( \| x_2, z \| + \| x_3, z \| \right) \]

\[ \leq K^2 \left( \| x_1, z \| + \| x_2, z \| + \| x_3, z \| \right) \]

\[ = K^{1+\lfloor \log_2(2-1) \rfloor} \left( \| x_1, z \| + \| x_2, z \| + \| x_3, z \| \right) , \]

\[ \| x_1 + x_2 + x_3 + x_4, z \| \leq K \left( \| x_1 + x_2 + x_3 + x_4, z \| \right) \]

\[ \leq K^3 \left( \| x_1, z \| + \| x_2, z \| + \| x_3, z \| + \| x_4, z \| \right) \]

\[ = K^{1+\lfloor \log_2(4-1) \rfloor} \left( \| x_1, z \| + \| x_2, z \| + \| x_3, z \| + \| x_4, z \| \right) . \]

The above inequalities actually mean that the inequality (12) holds true for \( m = 2, 3, 4 \). Let’s assume that (12) holds true for each positive integer, such that it is an element of \{2, 3, ..., 2^k, 2^{k+1} + 1, ..., 2^{k+1} \}. Let \( m \in \{2^{k+1} + 1, 2^{k+1} + 2, ..., 2^{k+2}\} \). Then, it exists \( p, q \in \{2^k, 2^k + 1, 2^{k+1} + 2, ..., 2^{k+1}\} \) so that \( p + q = m \), thus

\[ \| x_1 + x_2 + ... + x_m, z \| \leq K \left( \| x_1 + ... + x_p, z \| + K \| x_{p+1} + ... + x_m, z \| \right) \]

\[ \leq K \cdot K^{1+\lfloor \log_2(p-1) \rfloor} \left( \| x_1, z \| + \| x_2, z \| + ... + \| x_p, z \| \right) \]

\[ + K \cdot K^{1+\lfloor \log_2(q-1) \rfloor} \left( \| x_{p+1}, z \| + \| x_{p+2}, z \| + ... + \| x_m, z \| \right) \]

\[ \leq K \cdot K^{\lfloor \log_2(n-1) \rfloor} \left( \| x_1, z \| + \| x_2, z \| + ... + \| x_p, z \| \right) \]

\[ + K \cdot K^{\lfloor \log_2(n-1) \rfloor} \left( \| x_{p+1}, z \| + \| x_{p+2}, z \| + ... + \| x_m, z \| \right) \]

\[ = K^{1+\lfloor \log_2(n-1) \rfloor} \left( \| x_1, z \| + \| x_2, z \| + ... + \| x_{m-1}, z \| + \| x_m, z \| \right) . \]

By applying the principle of mathematical induction, we get that the inequality (12) holds true for each \( n > 1 \) and for all \( x_1, x_2, ..., x_n \in L \). ■

**Theorem 10.** Let \( L \) be a quasi 2-normed space with modulus of concavity
$K \geq 1$ and $n > 2$. For all $z, x_1, x_2, ..., x_n \in L$ such that $\|x_1, z\| \geq \|x_2, z\| \geq ... \geq \|x_n, z\| > 0$ the following inequalities hold true

$$K^{2+\lfloor \log_2(n-2) \rfloor} \sum_{i=1}^{n} \|x_i, z\| \leq K \|\sum_{i=1}^{n} x_i, z\| + [nK^{2+\lfloor \log_2(n-2) \rfloor}] - \|\sum_{i=1}^{n} \frac{x_i}{\|x_i, z\|}, z\| \|x_1, z\|, \quad (13)$$

$$\|\sum_{i=1}^{n} x_i, z\| + (nK^{2+\lfloor \log_2(n-2) \rfloor}) - K \|\sum_{i=1}^{n} \frac{x_i}{\|x_i, z\|}, z\| \|x_n, z\| \leq K^{2+\lfloor \log_2(n-2) \rfloor} \sum_{i=1}^{n} \|x_i, z\|. \quad (14)$$

**Proof.** Let $n > 3$. The properties of quasi 2-norm and Lemma 1, imply that

$$\|\sum_{i=1}^{n} x_i, z\| = \|\sum_{i=1}^{n} x_i, z\| + \left(\sum_{i=1}^{n} \frac{x_i}{\|x_i, z\|}, z\right), \|\sum_{i=1}^{n} x_i, z\| \leq K \|\sum_{i=1}^{n} \frac{x_i}{\|x_i, z\|}, z\| + K \|\sum_{i=1}^{n} \frac{x_i}{\|x_i, z\|}, z\| - \sum_{i=1}^{n} \frac{x_i}{\|x_i, z\|}, z\|$$

$$= \frac{K}{\|x_1, z\|} \|\sum_{i=1}^{n} x_i, z\| + \frac{K}{\|x_1, z\|} \|\sum_{i=2}^{n} \frac{x_i}{\|x_i, z\|}, z\|$$

$$\leq \frac{K}{\|x_1, z\|} \|\sum_{i=1}^{n} x_i, z\| + \frac{K^{1+\lfloor \log_2(n-2) \rfloor}}{\|x_1, z\|} \sum_{i=2}^{n} (\|x_1, z\| - \|x_i, z\|)$$

$$= \frac{K}{\|x_1, z\|} \|\sum_{i=1}^{n} x_i, z\| + \frac{K^{2+\lfloor \log_2(n-2) \rfloor}}{\|x_1, z\|} \left[n \|x_1, z\| - \sum_{i=1}^{n} \|x_i, z\|\right],$$

i.e. the following inequality holds true

$$\|\sum_{i=1}^{n} x_i, z\| \leq \frac{K}{\|x_1, z\|} \|\sum_{i=1}^{n} x_i, z\| + \frac{K^{2+\lfloor \log_2(n-2) \rfloor}}{\|x_1, z\|} \left[n \|x_1, z\| - \sum_{i=1}^{n} \|x_i, z\|\right].$$

The latter is equivalent to the inequality (13).

Let $n > 3$. Since the properties of quasi 2-norm and Lemma 1, it is true that

$$\|\sum_{i=1}^{n} \frac{x_i}{\|x_i, z\|}, z\| = \|\sum_{i=1}^{n} \frac{x_i}{\|x_i, z\|}, z\| + \left(\sum_{i=1}^{n} \frac{x_i}{\|x_i, z\|}, z\right), \|\sum_{i=1}^{n} \frac{x_i}{\|x_i, z\|}, z\| \leq K \|\sum_{i=1}^{n} \frac{x_i}{\|x_i, z\|}, z\| + K \|\sum_{i=1}^{n} \frac{x_i}{\|x_i, z\|}, z\| - \sum_{i=1}^{n} \frac{x_i}{\|x_i, z\|}, z\|$$

$$= K \|\sum_{i=1}^{n} \frac{x_i}{\|x_i, z\|}, z\| + \frac{K}{\|x_1, z\|} \sum_{i=1}^{n} \frac{\|x_i\|}{\|x_i, z\|}, x_i, z\|.$$
A remark about quasi 2-normed space

The following inequality holds true

\[ \left\| \sum_{i=1}^{n} \frac{x_i}{\|x_i, z\|}, z \right\| \leq K \left( \sum_{i=1}^{n-1} \left( \|x_i, z\| - \|x_i, z\| \right) \right) + K^{1+\log_2(n-2)} \sum_{i=1}^{n-1} \left( \|x_i, z\| - \|x_i, z\| \right), \]

i.e. the following inequality holds true

\[ \left\| \sum_{i=1}^{n} \frac{x_i}{\|x_i, z\|}, z \right\| \leq K \left( \sum_{i=1}^{n-1} \left( \|x_i, z\| - \|x_i, z\| \right) \right) + K^{2+\log_2(n-2)} \left( \sum_{i=1}^{n-1} \left( \|x_i, z\| - \|x_i, z\| \right) \right), \]

The latter is equivalent to the inequality (14). ■

References


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