Stability Analysis of a Host-Parasitoid Model with Logistic Growth Using Allee Effect

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Abstract

Allee effect and parasitism are common biological phenomena observed in nature. In this paper, we consider a discrete-time Host-Parasitoid model with and without Allee effect. We list all the possible equilibrium points of our Host-Parasitoid model. And we analyze the conditions under which our model is stable or unstable in both the cases.

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1 Introduction

The Allee effect, or the decrease in population growth rates at low population densities, was first described in the 1930s. The presence of Allee effect indicates that there is a minimal population size necessary for a population to maintain itself in nature. We use a system of difference equations to study the host-parasitoid interactions. Basically, we have three events in the life cycle of hosts and parasitoids, namely, density dependent competition in host populations, reproduction of hosts and parasitism. Depending on the order of these events, host-parasitoid systems are divided into two types. These types are distinguishable by considering whether density dependence in host population acts on hosts after parasitism or before parasitism. The framework of the models of these types are

\[ H_{t+1} = \lambda H_t g[H_t, f(P_t)] f(P_t) \]
\[ P_{t+1} = b H_t (1 - f(P_t)) \]  \hspace{1cm} (1)
\[ \lambda > 1, \ b > 0. \]

\[ H_{t+1} = \lambda H_t g[H_t] f(P_t) \]
\[ P_{t+1} = b H_t g[H_t](1 - f(P_t)) \]  \hspace{1cm} (2)
\[ \lambda > 1, \ b > 0. \]

Here

\( H_t, P_t \) - Population density of the hosts and parasitoids at generation \( t \).

\( \lambda \) - Intrinsic rate of natural increase of hosts.

\( b \) - Number of parasitoids which develop from one parasitized host.

Further, the functions \( g \) and \( f \) define the density dependence in host population growth and the fraction of hosts which can escape from parasitism, respectively [3], [4].

Allee effect in a discrete-time predator-prey system was studied by Canan Celik and Oktay Duman [1]. A host-parasitoid interaction with Allee effects on the host was studied by Sophia R.J. Jang, Sandra L. Diamond [7]. In this paper, we propose a host-parasitoid model and analyze its stability. Further, we analyze the stability of the host-parasitoid model with Allee effect. The Preliminaries are discussed in section 2. In section 3, we propose a Host-Parasitoid model with logistic growth. In section 4, we analyze the dynamic behaviour of the model. In section 5, we introduce the Allee effect into our model.
2 Preliminaries

Definition 2.1. An equilibrium point \((H^*, P^*)\) is said to be stable if for every \(\epsilon > 0\) there exists \(\delta > 0\) such that for every initial condition \((H_0, P_0)\), \(\|(H_0, P_0) - (H^*, P^*)\| < \delta\) implies \(\|(H_t, P_t) - (H^*, P^*)\| < \epsilon\) for all \(n > 0\). It is said to be unstable, if it is not stable.

Definition 2.2. An equilibrium point \((H^*, P^*)\) is said to be asymptotically stable if there exists \(\eta > 0\) such that , \(\|(H_0, P_0) - (H^*, P^*)\| < \eta\) and \((H_t, P_t) \to (H^*, P^*)\) as \(n \to \infty\).

3 A Host-Parasitoid Model with Logistic Growth

In this paper, we consider the following host-parasitoid model that does not belong to the previous two types.

\[ H_{t+1} = \lambda H_t g[H_t] f(P_t) \]
\[ P_{t+1} = b H_t (1 - f(P_t)) \]  \hfill (3)

\( \lambda > 1, b > 0. \)

Let us consider the intrinsic rate of the natural increase of hosts,

\[ \lambda = \exp(r(1 - \frac{H_t}{K})) \]  \hfill (4)

where \(r\) is the intrinsic growth rate, and \(K\) is the carrying capacity of the environment. Therefore, our model becomes

\[ H_{t+1} = \exp(r(1 - \frac{H_t}{K})) H_t g[H_t] f(P_t) \]
\[ P_{t+1} = b H_t (1 - f(P_t)) \]  \hfill (5)

\( b > 0. \)

Let us consider the single species model given by \(H_{t+1} = \lambda H_t g[H_t] = F(H_t)\) The functions \(g\) and \(f\) are assumed to satisfy \(H_1\) and \(H_2\), respectively:

\(H_1: F\) is a continuous function with \(F(0) = 0\) and there is an unique positive equilibrium point \(H^*\) such that \(F(H^*) = H^*, F(H) > H\) for \(0 < H < H^*\) and \(0 < F(H) < H\) for \(H^* < H\). Further, if \(F\) has a maximum at \(H_M \in (0, H^*)\), then \(F\) is monotonically decreasing for all \(H > H_M\).

\(H_2: f\) is a continuous function with \(f(0) = 1\) and \(0 < f(P) \leq 1\) for all \(P \geq 0\). The following functions are often employed as the functions \(f\) and \(g\), which satisfies \(H_2\) and \(H_1\) respectively, \(g\) is considered as the density dependence function: \(f(P_t) = \exp[-aP_t]\), \(g(H_t) = \exp[-\mu H_t]\), where \(a, \mu\) are positive constants. So finally, our model becomes \([2], [6], [7]\).
The Jacobian matrix is given by,
\[ J_{t+1} = H_t \exp \left( r \left( 1 - \frac{H_t}{K} \right) \right) \left( \exp[-\mu H_t] \exp[-aP_t] \right) \]
\[ P_{t+1} = bH_t(1 - \exp[-aP_t]) \] (6)
\[ b > 0. \]

4 Dynamic Behavior of the Model

In this section, we investigate the local behaviour of our model around each equilibrium point. The stability analysis of our model can be studied by computing the variation matrix corresponding to each fixed point \([3], [8]\). The variation matrix of the model at the state variable is given by,
\[ H_{t+1} = F_1(H_1, P_1) = H_t \exp \left( r \left( 1 - \frac{H_t}{K} \right) \right) \left( \exp[-\mu H_t] \exp[-aP_t] \right) \]
\[ P_{t+1} = F_2(H_2, P_2) = bH_t(1 - \exp[-aP_t]) \]

The Jacobian matrix is given by,
\[ J_{(H,P)} = \begin{pmatrix}
\frac{\partial F_1}{\partial H} & \frac{\partial F_1}{\partial P} \\
\frac{\partial F_2}{\partial H} & \frac{\partial F_2}{\partial P}
\end{pmatrix} = \begin{pmatrix}
r(1-aP_t) & -\mu H_t - aP_t \\
1 & -\mu H_t - aP_t
\end{pmatrix} \]

We list all the possible equilibrium points.

i) \( E_0 = (0, 0) \)
ii) \( E_1 = \left( \frac{Kr}{r+\mu K}, 0 \right) \)
iii) \( E_2 = \left( \frac{Kr}{r+\mu K}, \frac{rK-H^*(r+K\mu)}{aK} \right) \)

**Proposition 4.1.** The equilibrium point \( E_0 \) is locally asymptotically stable if \( e^r < 1 \), otherwise unstable.

**Proof.** Consider the Jacobian matrix \( J \) at \( E_0 \),
\[ J(E_0) = \begin{pmatrix}
e^r & 0 \\
0 & 0
\end{pmatrix} \] (7)

Hence, the eigenvalues of \( J \) at \( E_0 \) are \( Q_1 = e^r \) and \( Q_2 = 0 \). Thus it is stable, if \( e^r < 1 \). Otherwise, it is unstable. \( \square \)

**Proposition 4.2.** The equilibrium point \( E_1 \) is stable if it satisfies the conditions \( Q_1 < 1 \) and \( Q_2 < 1 \), otherwise unstable.

**Proof.** In order to prove the result, consider the Jacobian matrix at \( E_1 \)
\[ J_{(E_1)} = \begin{pmatrix}
\frac{\partial F_1}{\partial H} & \frac{\partial F_1}{\partial P} \\
\frac{\partial F_2}{\partial H} & \frac{\partial F_2}{\partial P}
\end{pmatrix} = \begin{pmatrix}
r(1-aP_t) & -\mu H_t - aP_t \\
1 & -\mu H_t - aP_t
\end{pmatrix} \]

Consider \( |J(E_1) - QI| = 0 \)
\[ Q^2 - (\left| e^{r(1-\frac{r}{r+\mu K})-\mu \frac{Kr}{r+K\mu} \left[ 1 + \frac{Kr}{r+K\mu} \left( 1 - \frac{r}{r+K\mu} \right) - \mu \right] \right| ab \frac{Kr}{r+K\mu} Q + \left( e^{r(1-\frac{r}{r+\mu K})-\mu \frac{Kr}{r+K\mu} \left[ 1 + \frac{Kr}{r+K\mu} \left( 1 - \frac{r}{r+K\mu} \right) - \mu \right] ab \frac{Kr}{r+K\mu} \right) = 0 \]
The roots of the above given characteristic equation are

\[ Q_1 = e^{\left[ r\left(1 - \frac{r}{r+K^*_r}\right) - \mu \frac{K^*_r}{r+K^*_r} \right]} \left[ 1 + \frac{K^*_r}{r+K^*_r} \left( \frac{r}{r+K^*_r} - \mu \right) \right], \quad Q_2 = ab^*_r \frac{K^*_r}{r+K^*_r}. \]

Therefore, the fixed point \( E_1 \) is stable if \( e^{\left[ r\left(1 - \frac{r}{r+K^*_r}\right) - \mu \frac{K^*_r}{r+K^*_r} \right]} < \left( \frac{1}{1 + \frac{K^*_r}{r+K^*_r} \left( \frac{r}{r+K^*_r} - \mu \right)} \right) \) and \( \frac{K^*_r}{r+K^*_r} < \left( \frac{1}{ab^*_r} \right). \) Otherwise it is unstable.

\[ \square \]

Consider the jacobian matrix at \( E_2 \)

\[ J(E_2) = \begin{pmatrix} \frac{e^{\left[ r\left(1 - \frac{r}{r+K^*_r}\right) - \mu H^* - aP^* \right]} \left[ 1 + H^* \left( \frac{r}{r+K^*_r} - \mu \right) \right]}{b(1-e^{-aP^*})} & e^{\left[ r\left(1 - \frac{r}{r+K^*_r}\right) - \mu H^* - aP^* \right]}[-aH^*] \\ ab^*_r & \end{pmatrix} \]

The characteristic equation is

\[ P_2(Q) = Q^2 - \text{tr} J(E_2)Q + \text{det} J(E_2) \Rightarrow Q^2 - e^{\left[ r\left(1 - \frac{r}{r+K^*_r}\right) - \mu H^* - aP^* \right]} \left[ 1 + H^* \left( \frac{r}{r+K^*_r} - \mu \right) \right] Q - abH^* + e^{\left[ r\left(1 - \frac{r}{r+K^*_r}\right) - \mu H^* - aP^* \right]} \left[ 1 + H^* \left( \frac{r}{r+K^*_r} - \mu \right) \right] abH^* \\
+ e^{\left[ r\left(1 - \frac{r}{r+K^*_r}\right) - \mu H^* - aP^* \right]}[aH^*])b(1-e^{-aP^*}) = 0 \]

It follows from the jury’s conditions \([5]\), that the modulus of all the roots of the above characteristic equation is less than 1, if and only if the conditions \( P_2(1) > 0, P_2(-1) > 0 \) and \( \text{det} J(E_2) < 1 \) hold. We first obtain that, \( P_2(1) > 0 \) holds for any

\[ e^{\left[ r\left(1 - \frac{r}{r+K^*_r}\right) - \mu H^* - aP^* \right]} < \left( \frac{1}{1 + H^* \left( \frac{r}{r+K^*_r} - \mu \right) \left[ 1 - abH^* - abH^* (1-e^{-aP^*}) \right]} \right) \]

On the other hand, \( P_2(-1) > 0 \) holds if and only if

\[ e^{\left[ r\left(1 - \frac{r}{r+K^*_r}\right) - \mu H^* - aP^* \right]} > \frac{-abH^* - 1}{\left[ 1 + H^* \left( \frac{r}{r+K^*_r} - \mu \right) \right] [1 + abH^*] + abH^* (1-e^{-aP^*})} \]

Finally, \( \text{det} J(E_2) < 1 \) holds if and only

\[ e^{\left[ r\left(1 - \frac{r}{r+K^*_r}\right) - \mu H^* - aP^* \right]} < \frac{1}{\left[ 1 + H^* \left( \frac{r}{r+K^*_r} - \mu \right) \right] [abH^*] + abH^* (1-e^{-aP^*})} \]

\[ \text{Theorem 4.3. The positive equilibrium} \ (H^*, P^*) \ \text{of the host-parasitoid system} \ (6) \ \text{is asymptotically stable, if} \]

\[ \frac{-abH^* - 1}{(A + abH^*)} < e^{\left[ r\left(1 - \frac{r}{r+K^*_r}\right) - \mu H^* - aP^* \right]} \left( \frac{1}{(A + abH^*)} \right) \]

holds, where \( A = \left[ 1 + H^* \left( \frac{r}{r+K^*_r} - \mu \right) \right], B = abH^* (1-e^{-aP^*}) \).

The next result follows from Theorem 4.3 immediately \([1], [5]\).
Corollary 4.4. The positive equilibrium \((H^*, P^*)\) of the model (6) is unstable if and only if

\[
\frac{-abH^* - 1}{(A[1 + abH^*] + B)} > e^{r(1 - \frac{H^*}{K}) - \mu H^* - aP^*}
\]

(or)

\[
e^{r(1 - \frac{H^*}{K}) - \mu H^* - aP^*} > \frac{1}{(A[abH^*] + B)}
\]

holds, where \(A = [1 + H^* \left(\frac{r}{K} \left(1 - \frac{H^*}{K}\right) - \mu\right)]\), \(B = abH^* \left(1 - e^{-aP^*}\right)\).

5 A Host Parasitoid Model with Allee Effect on the Host

Suppose, now that the host population experiences Allee effect but not the parasitoid population. We let \(\frac{H - c}{M + H}\) be the probability of a host individual successfully finding a mate or a cooperative individual, when the host population is of size \(H\). The parameter \(c\) is the lower bound for the host and \(M\) can be defined as the Allee effect constant [6], [7]. The host-parasitoid interaction now takes the following form:

\[
\begin{align*}
H_{t+1} &= H_t \exp(r(1 - \frac{H_t}{K}) \frac{H_t - c}{M + H_t}) \exp(-\mu H_t) \exp(-aP_t) \\
P_{t+1} &= bH_t (1 - \exp(-aP_t))
\end{align*}
\]

\(b > 0, H_0, P_0 > 0\).

where we take \(\frac{H_t - c}{M + H_t}\) as the Allee effect function. Where \(0 < M < \frac{r(K - H^*) (H^* - c) - \mu H^* K}{\mu H^* K}\).

The equilibrium points of the new system (13) are as follows \(E_0^* = (0, 0)\), \(E_1^* = \left(\frac{r_H}{\frac{K}{2} \left(\frac{K}{H^*} - \frac{c}{M + H^*}\right)}, 0\right)\) and \(E_2^* = \left(\frac{P^K}{a(1 - e^{-aP^*})}, \frac{K - H^*}{\frac{K}{2} \left(\frac{K}{H^*} - \frac{c}{M + H^*}\right)} - \frac{\mu H^*}{a}\right)\).

Consider the jacobian matrix \(J_1\) of (13) at \(E_0^*\)

\[
J_1(E_0^*) = \begin{pmatrix}
    e^{-\frac{r^*}{K}} & 0 \\
    0 & 0
\end{pmatrix}
\]

The eigen values of the corresponding matrix are \(Q_1 = e^{-\frac{r^*}{K}}\) and \(Q_2 = 0\).

Then the equilibrium point \(E_0^*\) is stable when \(e^{-\frac{r^*}{K}} < 1\), otherwise unstable. Similarly consider the jacobian matrix \(J_1\) at \(E_1^*\),

\[
J_1(E_1^*) = \begin{pmatrix}
    e^{r(1 - \frac{H^*}{K}) \frac{H^* - c}{M + H^*} - \mu} (1 - H^* \left(\frac{r}{K} \left(1 - \frac{H^*}{K}\right) - \mu\right) + \mu) - aH^* e^{r(1 - \frac{H^*}{K}) \frac{H^* - c}{M + H^*} - \mu} \\
    0
\end{pmatrix}
\]

The characteristic equation of the above matrix is given by...
\[ Q^2 - tr J_1(E_1^*) Q + det J_1(E_1^*) = 0 \]

Where \( tr J_1(E_1^*) = e^{(1 - \frac{H^*}{K}) (\frac{H^*}{K} + \mu)} (1 - H^* (\frac{r}{K} (1 - \frac{H^*}{K}) - \frac{M + c}{(M + H^*)^2} + \mu)) \) + \( abH^* \)

\[ det J_1(E_1^*) = e^{(1 - \frac{H^*}{K}) (\frac{H^*}{K} + \mu)} (1 - H^* (\frac{r}{K} (1 - \frac{H^*}{K}) - \frac{M + c}{(M + H^*)^2} + \mu)) abH^* \]

The eigen values are as follows,

\[ Q_1 = e^{(1 - \frac{H^*}{K}) (\frac{H^*}{K} + \mu)} (1 - H^* (\frac{r}{K} (1 - \frac{H^*}{K}) - \frac{M + c}{(M + H^*)^2} + \mu)), \]

\[ Q_2 = abH^* \]

Therefore, the equilibrium point \( E_1^* \) is stable, if

\[ e^{(1 - \frac{H^*}{K}) (\frac{H^*}{K} + \mu)} < \frac{1}{(1 - H^* (\frac{r}{K} (1 - \frac{H^*}{K}) - \frac{M + c}{(M + H^*)^2} + \mu))} \]

and \( H^* < (\frac{1}{ab}) \). Otherwise it is unstable.

Consider the jacobian matrix \( J_1 \) at \( E_2^* \),

\[ J_1(E_2^*) = \left( \left( \frac{H^*}{K} + \frac{N}{M + H^*} \right) (1 - H^* (\frac{r}{K} (1 - \frac{H^*}{K}) - \frac{M + c}{(M + H^*)^2} + \mu)) \right) \]

The characteristic equation is given by

\[ P_2^* (Q) = Q^2 - e^{(1 - \frac{H^*}{K}) (\frac{H^*}{K} + \mu)} (1 - H^* (\frac{r}{K} (1 - \frac{H^*}{K}) - \frac{M + c}{(M + H^*)^2} + \mu)) Q \]

\[ -abH^* e^{-aP^*} + e^{(1 - \frac{H^*}{K}) (\frac{H^*}{K} + \mu)} (1 - H^* (\frac{r}{K} (1 - \frac{H^*}{K}) - \frac{M + c}{(M + H^*)^2} + \mu)) \]

\[ \left( abH^* e^{-aP^*} + aH^* e^{(1 - \frac{H^*}{K}) (\frac{H^*}{K} + \mu)} \right) \]

\[ b \left( 1 - e^{-aP^*} \right) = 0 \] (15)

We consider the jury’s conditions[5] and we first obtain that \( P_2^* (1) > 0 \) holds iff

\[ e^{(1 - \frac{H^*}{K}) (\frac{H^*}{K} + \mu)} < \left( \left( 1 - H^* (\frac{r}{K} (1 - \frac{H^*}{K}) - \frac{M + c}{(M + H^*)^2} + \mu) \right) \right) \]

\[ \left( 1 - abH^* e^{-aP^*} \right) \]

where \( 0 < M < \frac{r(K - H^*)(H^* - c) - \mu H^* K}{\mu H^* K} \).

The condition \( P_2^* (-1) < 0 \) holds iff

\[ e^{(1 - \frac{H^*}{K}) (\frac{H^*}{K} + \mu)} > \left( \left( 1 - H^* (\frac{r}{K} (1 - \frac{H^*}{K}) - \frac{M + c}{(M + H^*)^2} + \mu) \right) \right) \]

\[ \left( 1 + abH^* e^{-aP^*} \right) \]

We obtain \( det J_1(E_2^*) < 1 \), if and only if

\[ e^{(1 - \frac{H^*}{K}) (\frac{H^*}{K} + \mu)} < \left( \left( 1 - H^* (\frac{r}{K} (1 - \frac{H^*}{K}) - \frac{M + c}{(M + H^*)^2} + \mu) \right) \right) \]

\[ \left( 1 + abH^* e^{-aP^*} \right) \]

**Theorem 5.1.** The positive equilibrium \((H^*, P^*)\) of the host-parasitoid system (13) is asymptotically stable, if

\[ \left( \frac{-1 - F}{D(1 + F) + abH^*(1 - e^{-aP^*})} \right) < e^{(1 - \frac{H^*}{K}) (\frac{H^*}{K} + \mu)} < \left( \frac{1}{D + abH^* F} \right) \]

holds.

Where \( D = (1 - H^* (\frac{r}{K} (1 - \frac{H^*}{K}) - \frac{M + c}{(M + H^*)^2} + \mu)), \)

\( F = abH^* e^{-aP^*} \).
The following result is an immediate consequence of Theorem 5.1.

**Corollary 5.2.** The positive equilibrium \((H^*, P^*)\) of the model (13) is unstable if and only if

\[
-1 - \frac{F}{D(1 + F) + abH^*(1 - e^{-aP^*})} > e^{r(1 - \frac{H^*}{K})(\frac{H^* - e}{M + H^*} - \mu H^* - aP^*)}
\]

(or)

\[
e^{r(1 - \frac{H^*}{K})(\frac{H^* - e}{M + H^*} - \mu H^* - aP^*)} > \frac{1}{[DF + abH^* - F]}
\]

holds, where

\[
D = \left(1 - H^* \left(\frac{r}{K} - \frac{H^*}{K} - \frac{M + e}{(M + H^*)^2} + \mu\right)\right), \ F = abH^*e^{-aP^*}.
\]

### 6 Conclusion

In this paper, we have proposed a host-parasitoid model with logistic growth. We have analyzed the stability of the model (6) without Allee effects and we have found some conditions in which the equilibrium points \(E_0, E_1\) and \(E_2\) are stable or unstable. Ecological facts lead us to the importance of Allee effect in population dynamics. In this paper, we have studied such an effect on the Host population. By mathematical analysis, we have shown the impact of the Allee effect (on the host population) on the stability of the positive equilibrium point for a discrete-time host-parasitoid model. That is, we have derived some conditions for which the equilibrium points \(E_0^*, E_1^*\) and \(E_2^*\) are stable or unstable. Thus, we have analyzed the host-parasitoid model with and without Allee effect.

### References


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