On Khalimsky Topology and Applications on the Digital Image Segmentation

M. Al Hajri
University of Dammam
College of Science, Department of Mathematics
P.O.Box 383, 31113, Dammam, Saudi Arabia

K. Belaid
University of Dammam
College of Science, Department of Mathematics
P.O. Box 383, 31113, Dammam, Saudi Arabia

L. Jaafar Belaid
University of Dammam
College of Science, Department of Mathematics
P.O. Box 383, 31113, Dammam, Saudi Arabia

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Abstract
The main goal of this work is to deal with the Khalimsky digital topology and its application in segmentation. First, we begin by giving some theoretical results on Khalimsky topology, the one point compactification and separation axioms. Then, we present and discuss digital applications in imaging. More precisely, numerical results on segmentation, contours detection and skeletonization are proposed. We end this paper by some concluding remarks.

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1 Introduction

Different applications including medical imaging applications are based on the morphological analysis and processing of 2D and 3D images. In this context, the segmentation plays a very important role not only in the extraction of information, but also in pattern recognition. More precisely, the segmentation had a high success in medicine and biology fields since its existence. Interested readers can refer to [7, 16, 2]. On the other hand, algorithms using this technique are very interesting, in the sense that the principle of segmentation is essentially based on the representation of forms in a simplified way with a total preservation of the topological information. On the other part, the skeletonization derives from the segmentation. Its concept allows to reduce considerably the size of storing images and this represents a real potential for data compression, concentration information, and computation time. The notion of skeletonization was introduced by H. Blum [7] in the sixties, by giving the following informal definition: Let $F$ be a homogeneous dry grass field, and let $\Omega$ be a set of points including in $F$. Let us suppose that all the contour points of $\Omega$ are ignited simultaneously such that the fire extends across $F$ at a constant speed, then the skeleton of $\Omega$ is defined as the locus of points where the burning fronts met. However, in formal terms, the notion of skeleton has a topological meaning using the principle of maximal balls and an equivalence of the formal and informal definitions was proved by L. Calabi and W.E. Hartnett in 1968, [8]. Our motivation in this work is based on the three following facts: first, the skeletonization seems not to have been used in the theoretical field and its contribution to the topology does not seem being analyzed. Second, different methods in segmentation have been developed but those based on the skeletonization principle do not seem to be fairly explored. Third, several studies have been made in spectral topology, more precisely on the compactification, however, to our knowledge, no application has been made in this area. To better explore the previous facts, some works should be cited: In 2009, D. Auroux [3], has given an application of the topological asymptotic expansion to the medical image segmentation problem. More precisely, the author has generalized the topological gradient approach used in image restoration and edge detection, proposed by L. Jaafar Belaid et al. in 2006 [10], for the segmentation problem and gave different applications to medical imaging. Next, in 2010, L. Jaafar Belaid gave some inconveniences of the use of the topological gradient in the segmentation problem [12]. More precisely, the fact that, the topological gradient approach do not give necessary closed contours make the results of the segmentation process not suitable enough. On the other part, from 2004 to 2013 K. Belaid et al. [4, 5, 6, 1] have studied some topological spaces and compactifications. More precisely, let $Y$ be a topological space and let $X$ be a subset of $Y$. $Y$ is said to be a compactification of $X$, if the following
properties are verified:

- $Y$ is compact,
- $X$ is dense subset of $Y$, and $Y \setminus X$ is called the reminder of $X$ in $Y$.

This means that in order to obtain $Y$ compact with $X$ a given subset of $Y$, it suffices to add $Y \setminus X$ to $X$. Now, in the discrete case, by considering digital images, we want to add at least a point in order to close the contours detected. This is can be defined as a discretization of the compactification process.

The structure of this work is the following. Section 2 is devoted to the Khalimsky topology. We give in this section theoretical results on the Khalimsky plane including separation axioms and the one point compactification. We propose in section 3 to study some applications in digital imaging. First, we review some preliminary notions on binary digital images, which will be used in this paper. Then, we propose numerical applications on some classical problems in imaging. More precisely, a contour detection and a segmentation result are discussed in this part. Numerical results are presented and some numerical comparisons with the topology of Rosenfeld are also given in this section. Finally, we propose an application in skeletonization and interpret this result as a discrete case of the one point compactification. We end this paper by section 4 through some concluding remarks.

## 2 Khalimsky topology, the one point compactification and separation axioms

Let us first recall that a topological basis associated to the Khalimsky topology defined on $\mathbb{Z}$, is given by the set:

$$\{\{2n + 1\}; \{2n - 1, 2n, 2n + 1\}; n \in \mathbb{Z}\}. \quad (1)$$

This means that for the Khalimsky line, the open sets are given by $\{\{m - 1, m, m + 1\}; m = 2n + 1; n \in A\} \cup \{\{m\}; m = 2n; n \in B\}$ where $A, B \subseteq \mathbb{Z}$. So, the even points are closed. By considering the product topology, we easily deduce the definition of the Khalimsky plane $\mathbb{Z}^2$ for which the open set are the points $(2n + 1, 2n + 1); n \in \mathbb{Z}$ and so, the closed sets are the points $(2n, 2n); n \in \mathbb{Z}$. More precisely, the Khalimsky plane is equipped with topology $\mathcal{K}$ generated by: for each $P = (m, n) \in \mathbb{Z}^2$,

$$\begin{cases} 
\{P, (m - i, n + j), i, j \in \{-1, 0, 1\}\}, & \text{if both } m \text{ and } n \text{ are even} \\
\{P, (m + i, n), i \in \{-1, 1\}\}, & \text{if } m \text{ is even and } n \text{ is odd} \\
\{P, (m, n + j), j \in \{-1, 1\}\}, & \text{if } m \text{ is odd and } n \text{ is even} \\
\{P\}, & \text{otherwise.} 
\end{cases} \quad (2)$$
In order to define the one point compactification of the Khalimsky plane, we begin by recalling first the one point compactification construction of a non compact topological space $X$:

set $\bar{X} = X \cup \{\infty\}$ with the topology $\bar{T}$ whose members are the open subsets of $X$ and all subsets $U$ of $X$ such that $X - U$ is a closed compact subset of $X$. Then $(\bar{X}, \bar{T})$ is a compactification of $X$ called the one point compactification or the Alexandroff compactification of $X$.

So, we have the following definitions.

**Definition 2.1.**
1. The one point compactification of the Khalimsky line is called the infinite Khalimsky circle $(\mathbb{Z} \cup \{\infty\}, \mathcal{K})$.
2. The one point compactification of the Khalimsky plane $(\mathbb{Z}^2, \mathcal{K}^2)$ is denoted by $S^2_\infty = (\mathbb{Z}^2 \cup \{\infty\}, \mathcal{K}^2)$ is called the infinite Khalimsky sphere.

**Definition 2.2.** Let $X$ be a topological space.
1. A subset $S$ of $X$ is said to be irreducible, if every two non empty open sets of $S$ has a non empty intersection.
2. An element $x$ of $X$ is said to be a generic point of a closed set $F$ if $F = \{x\}$.
3. A space $X$ is said to be sober if any non empty irreducible closed set of $X$ has a unique generic point.

**Proposition 2.3.** The Khalimsky plane is sober.

**Proof.** Let $C$ be a closed irreducible set of $\mathbb{Z}^2$. Since $\{z\}$ is an open set, for each $z = (2n + 1, 2m + 1)$ with $n, m \in \mathbb{Z}$, $C$ has at most an element $z$ of the form $z = (2n + 1, 2m + 1)$. We discuss two cases:

1. Case 1: there exist $n, m \in \mathbb{Z}$ such $z = (2n + 1, 2m + 1) \in C$. Then $\{z\} \subseteq C$. Since $\{z\}$ is open and $C$ is irreducible, $C - \{z\} = \emptyset$. So that $C = \{z\}$.
2. Case 2: for each $n, m \in \mathbb{Z}$ such $z = (2n + 1, 2m + 1) \notin C$. Then, there exists $x = (p, q)$ such that either $p$ or $q$ is even such that $x \in C$. Hence $\{x\}$ is an open set of $C$. Thus $C = \{x\}$.

Therefore $\mathbb{Z}^2$ is sober. \qed

Recall that a topological space is said to be spectral if it is homeomorphic to the spectrum of a ring equipped with the Zariski topology. In [4], K. Belaid et al. have proved that the Khalimsky line is not spectral but the infinite Khalimsky circle is spectral. It is immediate that the Khalimsky plane is not spectral, since it is not compact. The following result generalizes that of the paper [4].
Proposition 2.4. The infinite Khalimsky sphere is spectral.

Proof. The fact that \( \mathbb{Z}^2 \) is sober and has a basis of compact closed sets closed under finite intersection is immediate. Let \( F \) be a compact closed set \( F \) of \( X \) and \( O = \mathbb{Z}^2 - F \). Then \( O \) is co-compact and \( O \subseteq \mathbb{Z}^2 - F \). Since compact sets of the Khalimsky plane are finite, then \( U \cap O \) is compact for every compact open or closed set \( U \) of \( X \). So, by Theorem 2.2 of [4], we deduce that the one point compactification of the Khalimsky plane is spectral.

Remark 2.5. The Khalimsky line and Khalimsky plane are \( T_0 \) (that is, whenever \( x \) and \( y \) are distinct points of \( X \), there exists an open set which contains one of them but not the other) and are not \( T_1 \) (that is, \( \{ x \} = \{ x \} \), for each \( x \in X \)).

To study the separation axioms satisfied by the Khalimsky topology, let us recall that for a subset \( S \) of a topological space \( X \), the derived set \( \text{Der}(S) \) is the collection of accumulation points of \( S \). Hence for \( x \in X \), \( \text{Der}(x) = \overline{\{ x \}} - \{ x \} \). For two subsets \( A \) and \( B \) of \( X \), the notation \( A \not\vdash B \) is used to indicate that there exists an open set \( O \) of \( X \) such that \( A \subseteq O \) and \( O \cap B = \emptyset \).

Definition 2.6. A topological space \( X \) is called:

(i) \( T_D \) if for each \( x \in X \), \( \text{Der}(x) \) is closed.

(ii) \( T_{UD} \) if for each \( x \in X \), \( \text{Der}(x) \) is the union of disjoint closed sets.

(iii) \( T_{DD} \) if it is \( T_D \) and for each distinct elements \( x, y \) of \( X \), \( \text{Der}(x) \cap \text{Der}(y) = \emptyset \).

(iv) Weakly submaximal if each finite subset \( F \) of \( X \) is a locally closed set (that is, for each finite subset \( F \) of \( X \), \( \overline{F} - F \) is a closed set of \( X \)).

(v) \( T_F \) if for any point \( x \) and any finite subset \( F \) of \( X \) such that \( x \notin F \), either \( x \not\vdash F \) or \( F \not\vdash x \).

(vi) \( T_{FF} \) if for any two finite subsets \( F_1 \) and \( F_2 \) of \( X \) with \( F_1 \cap F_2 = \emptyset \), either \( F_1 \not\vdash F_2 \) or \( F_2 \not\vdash F_1 \).

(vii) \( T_Y \) if for each distinct \( x, y \in X \), \( \overline{\{ x \}} \cap \overline{\{ y \}} \) is either a singleton or the empty set.

(viii) \( T_{YS} \) if for each distinct \( x, y \in X \), \( \overline{\{ x \}} \cap \overline{\{ y \}} \) is either \( \emptyset \) or \( \{ x \} \) or \( \{ y \} \).

Classically we have the following implications:
Remark 2.7. The Khalimsky plane \((\mathbb{Z}^2, \mathcal{K})\) is not a \(T_{DD}\)-space, it is not a weakly submaximal space and it is not a \(T_F\)-space.

Proposition 2.8. Let \(X\) be a non-compact topological space. Then \(X\) is a \(T_D\)-space if and only if the one point compactification \(\tilde{X}\) of \(X\) is a \(T_D\)-space.

Proof. It is immediate that if \(X\) is a \(T_D\)-space then \(X\) is a \(T_D\)-space. Now let \(x \in \tilde{X}\). We consider two cases:

1. Case 1: \(x \in X\). Since \(X\) is a \(T_D\)-space, \(\{x\}\) is a locally closed set of \(X\). Hence there exists an open set \(O\) and a closed set \(C\) of \(X\) such that \(\{x\} = C \cap O\). Since \(C\) is a closed set of \(X\), \(C \cup \{\infty\}\) is a closed set of \(\tilde{X}\). Thus \(\{x\} = (C \cup \{\infty\}) \cap O\). Therefore \(\{x\}\) is a locally closed set of \(\tilde{X}\).

2. Case 2: \(x = \infty\). Then \(\{\infty\}\) is a closed set of \(\tilde{X}\).

\[\Box\]

Proposition 2.9. The Khalimsky plane \((\mathbb{Z}^2, \mathcal{K})\) and the Khalimsky sphere are \(T_D\)-spaces.

Proof. To prove that the Khalimsky plane \((\mathbb{Z}^2, \mathcal{K})\) is a \(T_D\)-space it is sufficient to remark that for any \(z = (x, y) \in \mathbb{Z}^2\)

- if \(x, y\) are even, \(\overline{\{z\}} = \{z\}\).
- if \(x\) is even and \(y\) is odd \(\overline{\{z\}} = \{(x, y + k) \mid k \in \{-1, 1\}\}\),
- if \(x\) is odd and \(y\) is even, \(\overline{\{z\}} = \{(x + k, y) \mid k \in \{-1, 1\}\}\),
- \(\overline{\{z\}} = \{(x - i, y + k) \mid i, k \in \{-1, 0, 1\}\}\) otherwise.

Now, using the above proposition, the Khalimsky sphere is \(T_D\).

\[\Box\]
3 Applications

Let us first recall that generally any image is considered as discrete and finite. This definition is based on two different approaches. The first one considers that images are continuous functions defined on \( \mathbb{R}^n \) but can be approached using for example interpolation methods. The second one supposes that the image is initially defined on a discrete space. This last approach can also be seen from two different points of view: the image is discrete because on the continuous space \( X \), one has considered a finite number of sensors. In this case, the image is considered as discrete because it has been structured on \( \mathbb{Z}^n \). Whereas digital images are initially defined on \( \mathbb{Z}^n \). This natural structure on \( \mathbb{Z}^n \) is then based on a grid or a mesh. The images considered in this paper are considered as digital images.

Different approaches have been proposed for the study of topological properties of binary digital images and results are more or less comparable. The most widely used are the Khalimsky topology [13] and the Rosenfeld topology [17]. The first one is an Alexandroff topology, for which any intersection of open sets is open. However, the second one does not define a topology. The elements are linked together using an adjacency relationship that defines connections between elements. This connectivity concept does not allow the construction of a topology.

This part is devoted to study the effect of the chosen topology defined on the space, on some applications for digital images.

3.1 Digital topology and segmentation

Let us consider a binary digital image \( I \) defined on a domain \( \Omega \subset \mathbb{Z}^2 \). Since \( I \) is digital then \( I \) is supposed to be finite with respect to a grid of the plane and is represented by a set of pixels \( (I_{ij}) \). We recall that a such image can be represented by a matrix which admits 1 and 0 as coefficients. 1 represents the white pixels and 0 represents the black ones. Each pixel can be related to its 4 neighborhood or 8 neighborhoods. Set, \( x = (x_1, x_2), y = (y_1, y_2) \) in \( \mathbb{Z}^2 \), we define the square distance and the diamond distance by

\[
d_4(x, y) = |y_1 - x_1| + |y_2 - x_2|
\]

and

\[
d_8(x, y) = \max (|y_1 - x_1|, |y_2 - x_2|).
\]

It is easy to see that \( d_4 \) and \( d_8 \) verify the symmetry, the positivity and the triangle inequality axioms.

Using distances \( d_4 \) and \( d_8 \), we define the neighborhoods of a pixel \( x \in \mathbb{Z}^2 \) as following

\[
N_4(x) = \{ y \in \mathbb{Z}^2; d_4(x, y) \leq 1 \}
\]

and

\[
N_8(x) = \{ y \in \mathbb{Z}^2; \max (|y_1 - x_1|, |y_2 - x_2|) \leq 1 \}.
\]
and
\[ N_8(x) = \{ y \in \mathbb{Z}^2; d_8(x, y) \leq 1 \} \]  \hspace{1cm} (6)

Then
\[ N_4^*(x) = N_4(x) \setminus \{ x \} \]  \hspace{1cm} (7)

and
\[ N_8^*(x) = N_8(x) \setminus \{ x \} \]  \hspace{1cm} (8)

define the n-neighbors of \( \mathbf{x} \) with \( n = 4 \) (respectively, \( n = 8 \)) and each element \( y \in N_4^*(x) \) (respectively, \( y \in N_8^*(x) \)) is said to be adjacent to \( x \).

Let us mention that an important detail in segmentation lies to closed contours. L. Jaafar et al. have discussed this question in [11] and have presented a numerical method in order to obtain closed contours, which leads to an acceptable result of segmentation. A such detail of closed contours can be also related to Jordan theorem in \( \mathbb{R}^2 \), [9]. More precisely, any continuous, simple and closed curve in \( \mathbb{R}^2 \) divides the plane into two disjoint regions: the interior of the concerned domain and its complementary. Since we manipulate digital images, then one should verify the extension of a such result in the space \( \mathbb{Z}^2 \) and its effect on segmentation results. Many researchers have studied the discrete version of the theorem of Jordan and a multitude of algorithms have been presented in order to solve the problem of segmentation of digital images. We are interested in particular to the definitions presented by Kong et al. in [14], that we propose to briefly recall in this paper.

**Definition 3.1.** A binary digital image is defined by the function \( I : \Omega_{(m,n)} \longrightarrow [0,1] \), where \( \Omega_{(m,n)} \subseteq \mathbb{Z}^2 \) and \( (m,n) \) defines the connectivity of the image \( I \).

Distances \( d_4 \) and \( d_8 \) defined by equations 3 and 4 together to the notion of neighborhoods given by equations 5 and 6, give the different cases of connections one can construct. Some questions arise from the connections obtained: which digital topology is the most appropriate to the discrete case of the theorem of Jordan? what results can be obtained in the segmentation process? what is the effect of the choice of the topology on closed contours? to better illustrate the interpretations of these questions, let us consider the 2 synthetic binary images given by Figure 1. The digital representation of the binary images 1 is given by Figure 2. In order to avoid any ambiguity, any digital image is defined by the data \( m, n, U, V \) where \( U \) is the domain of the discrete image (so it is exactly the set of all points of the discrete image \( I \)), \( W \) is the set of black points of \( I \) and \( m, n \) defines the connectivity of the binary image \( I \).

Any path \( p_i, 0 \leq i \leq n - 1 \) is a set of points such that \( p_i \) is adjacent to \( p_{i+1} \). The path is closed if \( p_0 = p_n \). (So, a unique point \( p_i \) can be considered as a particular closed path).

We recall that the discrete version of the Jordan theorem on \( \mathbb{Z}^2 \) is given by the following theorem, [17].
Figure 1: Examples of synthetic binary images.

Figure 2: Digital representation of the binary images given in Figure 1

Figure 3: An illustration of the 4 and the 8 connectivity.
Theorem 3.2. The complementary of any simple and closed curve of connectivity 4 (resp. of connectivity 8) admits 2 components of connectivity 8 (resp. of connectivity 4.)

Using the distances given by equations 3 and 4, together with the notion of adjacency given by the relations 5 and 6, we present in Figure 3 the different kind of connectivity that we can construct on the considered synthetic binary images. The first image (at left) in Figure 3, shows a closed curve of connectivity 4 given by the white pixels. The complementary of this curve forms a set of 2 connected components of connectivity 8. Conversely, the second image (at right), shows a closed curve of connectivity 4 given by the black pixels. Here, the white pixels are connected by an 8 connectivity and their complementary gives 2 connected components of connectivity 4. The discrete version of the Jordan theorem 3.2 is then verified.

Now, what about closed contours and segmentation results? in fact, from Figure 3, for both the 2 images, the black pixels are separated from the white ones by a closed curve. This separation makes successful the result of segmentation. The connections used here are \((m, n) = (4, 8)\) (respectively, \((m, n) = (8, 4)\)). It is clear that if a connection of type \((4, 4)\) is used (respectively. \((8, 8)\)), then we do not necessary obtain closed curves, which can be a major inconvenience for the segmentation process (see [11].)

Nor does it seem appropriate to use the Khalimsky topology in digital applications, especially for the segmentation process. More precisely, let us consider the initial image given by Figure 1 (at right). As the Khalimsky topology implies an 8 connection for both \((2n, 2n)\) and \((2n + 1, 2n + 1)\) pixels, and a 4 connection for the other pixels, then the connections between the different pixels of the initial image seen as a matrix \(5 \times 5\) is shown by Figure 4, which illustrates a portion of the Khalimsky digital plane.
Finally, we propose to illustrate the effect of these connections on the segmentation result. Figure 5 shows the contours obtained by both the Khalimsky topology (at right) and the Rosenfeld topology (at left). As it is shown, the results obtained using the Rosenfeld topology are better. First, the contours obtained using the Rosenfeld topology are closed and this optimizes the segmentation result. On the other hand, the Khalimsky topology does not necessary give closed contours. This is a major drawback in the segmentation process. Second, Figure 5 (at left) shows an over-segmentation at the corners in comparison with the result of segmentation obtained by the Rosenfeld topology. So, we conclude that the use of the Rosenfeld topology gives better results in segmentation.

3.2 Digital topology and squeletonization

Another point of view concerning the problem of over segmentation can be directly related to the process of squeletonization. More precisely, in order to assure an acceptable result of segmentation, we should make a particular interest to the number of connected components. To avoid this over-segmentation, we propose to use the method of elimination of simple points. We recall that the point $p$ is said to be simple if its removal does not affect the number of connected components of both a part $A$ and its complementary. A mathematical definition of such points can be found in [15] as follows.

**Definition 3.3.** Let $A \subset \mathbb{Z}^2$ and let $P \in A$. Then $P$ is said to be simple if the two following relations are satisfied:

$$
\begin{align*}
C\left([A \cap N_8^*(P)] \cup \{P\}\right) &= C\left([A \cap N_8^*(P)]\right), \\
C\left([A^c \cap N_8^*(P)] \cup \{P\}\right) &= C\left([A^c \cap N_8^*(P)]\right),
\end{align*}
$$

(9)
where $C(N(P))$ is the number of connected components of $N(P)$ and $A^c$ is the complementary of $A$.

**Remark 3.4.** Let us note that if $P \in A$ is simple then $C(A) = C(A \setminus \{P\})$ and $C(A^c) = C(A^c \cup \{P\})$.

Figure 6 shows the notion of simple points. Initially, the given region 6(a) presents a unique connected component (black points) and its complementary (white points) has 2 connected components. In 6(b), we can see the simple points (grey points) that can be removed. The result of skeletonization is then obtained after the elimination of simple points and is shown in 6(c). As we can remark it, the image conserves one black connected component and two white ones.

A classic method of skeletonization is based on an iterative approach. Let $X_1$ be the part of the image to be studied. If $x \in X_1 \subset I$ is not a simple point then we take another point $y \in X$, else if the point $x$ is eliminated and a new object $X_2$ is obtained. This iterative process is repeated until stability. The stability is attained when all simple points have disappeared. This approach gives the final result shown in Figure 6(c).

Moreover, suppose that, an error occurs and an additive point is eliminated from the final result of skeletonization (as it is shown by Figure 6(d)). First, the topology is not preserved (as this gives 2 black connected components and a unique white connected component). By adding this point, we obtain a compact set: this is exactly the notion of the one point compactification. Moreover, as discrete images are considered finite, then this process can be iterated as many times as necessary. We intend in a forthcoming paper to implement an algorithm for this purpose.

## 4 Conclusion

We have presented in this work a connection between theoretical results in topology and applied ones in digital imaging. We have presented some numerical results that show how the choice of the topology used can be crucial in preserving the desired properties of the initial image (topology, main contours, geometric characteristics, etc.). We aim in the future to implement some theoretical results and to take advantage in many applications, especially in medical ones where the main objective consists in splitting an image into its constituent parts, and automatically in identifying the several parts of the image studied as bones, tumor masses, cancer tissues, etc. These results can be very promising in applied mathematics and particularly, the results of the application of the compactification approach can be very interesting for future works.
Figure 6: Use of simple points in skeletonization
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