A Study for the Moment of Wishart Distribution

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Abstract

The skewness of a matrix quadratic form \( XX' \) is obtained using the expectation of stochastic matrix and applying the properties of commutation matrices, where \( X \sim \mathcal{N}_{p,n}(0, \Sigma, I_n) \).

Keywords: Commutation matrix, Moment, Wishart distribution

1 Introduction

Let \( p \times n \) stochastic matrix \( X \) be distributed as \( \mathcal{N}_{p,n}(0, \Sigma, \Phi) \), where \( \Sigma : p \times p \) and \( \Phi : n \times n \) are positive semidefinite, and \( E(X) = 0 \). Then, the variance-covariance matrix of the vectorization of \( X \) is \( \text{Cov}(\text{vec}X, \text{vec}X) = \Phi \otimes \Sigma \). For \( \Phi = I_n \), \( XX' \) is said to have Wishart distribution with scale matrix \( \Sigma \) and degrees of freedom parameter \( n \), where \( I_n \) is an \( n \times n \) identity matrix. Von Rosen (1988) has obtained moments of arbitrary order of matrix \( X \), and using these has calculated the second order moment of quadratic form \( XAX' \), where \( A : n \times n \) is an arbitrary non-stochastic matrix. Neudecker and Wansbeek (1987) have also obtained the second order moment of \( XAX' \) by calculating...
the expectation of $YAX'CYBY'$, where $Y = X + M$ is the matrix of means, and $A, B$ and $C$ are $n \times n$ arbitrary non-stochastic matrices. Tracy and Sultan (1993) obtained the third order moment of $XAX'$ using the sixth moment of matrix $X$. Kang and Kim (1996a) derived the vectorization of the general moment of $XAX'$. Also Kang and Kim (1996b) derived the vectorization of the general moment of non-central Wishart distribution, using the vectorization of the general moment of $XAX'$. We obtain the skewness of $XX'$ using the third moment of $XX'$.

Commutation matrices play a major role here. These and some useful results are presented in Section 2. The skewness of matrix quadratic form $XX'$ is obtained in Section 3.

## 2 Some Useful Results

**Definition 2.1** Let $A = (a_{ij})$ be an $m \times n$ matrix and $B = (b_{ij})$ be a $p \times q$ matrix. Then the Kronecker product $A \otimes B$ of $A$ and $B$ is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}.$$ 

**Definition 2.2** Let $A$ be an $m \times n$ matrix and $a_i$ the $i$th column of $A$; then vec$A$ is the $mn$ column vector

$$\text{vec}A = [a'_1, a'_2, \cdots, a'_n]' .$$

**Definition 2.3** Commutation matrix $I_{m,n}$ is an $mn \times mn$ matrix containing $mn$ blocks of order $m \times n$ such that $(ij)$th block has a 1 in its $(ji)$th position and zeroes elsewhere. One has

$$I_{m,n} = \sum_{i=1}^{n} \sum_{j=1}^{m} (H_{i,j} \otimes H'_{i,j})$$

where $H_{i,j}$ is an $n \times m$ matrix with a 1 in its $(ij)$th position and zeroes elsewhere, and can be written as $H_{i,j} = e_ie'_j$, where $e_i$ is the $i$th unit column vector of order $n$.

For an $m \times m$ identity matrix $I_m$ and $n \times n$ identity matrix $I_n$,

$$I_m \otimes I_n = I_{mn}, \quad (1)$$

$$I^{-1}_{m,n} = I'_{m,n} = I_{n,m}, \quad (2)$$
and

\[(I_n \otimes I_{n,n} \otimes I_n)(I_n \otimes I_n \otimes I_{n,n}) (I_n \otimes I_{n,n} \otimes I_n) = (I_n \otimes I_n \otimes I_{n,n})(I_n \otimes I_{n,n} \otimes I_n) (I_n \otimes I_n \otimes I_{n,n}).\]

For an \(m \times n\) matrix \(A\), \(I_{n,m} \text{vec} A = \text{vec} A'\),

\[(A \otimes B)' = A' \otimes B'.\]

For \(A, B, C,\) and \(D\) conformable matrices,

\[(A \otimes B) \otimes (C \otimes D) = (A \otimes B) + (A \otimes D) + (B \otimes C) + (B \otimes D).\]

For \(A\) and \(B\) conformable matrices,

\[\text{vec}' A' \text{vec} B = \text{tr} (AB).\]

For an \(m \times n\) matrix \(A\) and a \(p \times q\) matrix \(B\),

\[\text{vec} (A \otimes B) = (I_n \otimes I_{m,q} \otimes I_p)(\text{vec} A \otimes \text{vec} B).\]

For \(A, B,\) and \(C\) conformable matrices,

\[\text{vec}(ACB) = (B' \otimes A) \text{vec} C.\]

For \(A, B, C,\) and \(D\) conformable matrices,

\[(AB) \otimes (CD) = (A \otimes C)(B \otimes D).\]

For an \(m \times n\) matrix \(A\) and a \(p \times q\) matrix \(B\),

\[I_{m,p}(A \otimes B) = (B \otimes A) I_{n,q}.\]

For an \(m \times n\) matrix \(A\) and a \(p \times q\) matrix \(B\), and an \(r \times s\) matrix \(C\),

\[A \otimes B \otimes C = I_{r,m}p(C \otimes A \otimes B)I_{q,m,s} = I_{p,r,s}(B \otimes C \otimes A)I_{n,s,q}.\]

For an \(m \times n\) matrix \(A\) and a \(p \times q\) matrix \(B\), and an \(r \times s\) matrix \(C\),

\[\text{vec}(A \otimes B \otimes C) = (I_m \otimes I_{m,qs} \otimes I_{pr})(I_{mnq} \otimes I_{p,s} \otimes I_r)(\text{vec} A \otimes \text{vec} B \otimes \text{vec} C).\]

Balestra (1976) and Neudecker and Wansbeek (1983) discussed higher order commutation matrices, for which

\[I_{a,b,c} = (I_{a,c} \otimes I_b)(I_a \otimes I_{b,c}) = (I_{b,c} \otimes I_a)(I_b \otimes I_{a,c}),\]

\[I_{a,b,c} = (I_b \otimes I_{a,c})(I_{a,b} \otimes I_c) = (I_c \otimes I_{a,b})(I_{a,c} \otimes I_b).\]
\[ I_{n,n^2}I_{n,n^2} = I_{n^2,n} \quad \text{and} \quad I_{n^2,n}I_{n^2,n} = I_{n,n^2}. \]  

When \( A = I_n = \Phi \), \( E(XX') = n\Sigma \) where \( E(XX') \) is the first moment of \( XX' \) according to Magnus and Neudecker (1979).

Von Resen (1988) obtained the second moment of \( XX' \),
\[ E \left( \otimes^2 (XX') \right) = n(\vec{\Sigma} \vec{\Sigma}') + n^2 \left( \otimes^2 \Sigma \right) + nI_{p,p} \left( \otimes^2 \Sigma \right) . \]

### 3 Skewness of Wishart Distribution

**Notation**

\[ P(k,l;m) = I_{p^k,p^l} \otimes I_{p^m}, \]
\[ P(k;l,m) = I_{p^k} \otimes I_{p^l,p^m}, \]
\[ V = (\vec{\Sigma} \vec{\Sigma}') \otimes \Sigma, \]

and \( S_X = XX' \).

Tracy and Sultan (1993) gave the third moment of matrix quadratic form \( XX' \), in square matrix form,
\[ E \left( \otimes^3 S_X \right) = n^3 (\otimes^3 \Sigma) + n^2 V + n^2 Q (\otimes^3 \Sigma) + nQPVP + nPQ (\otimes^3 \Sigma) + n^2 QPV PQ + nQPV + nQPV (\otimes^3 \Sigma) + nPV + nPV PQ + n^2 PV P + n^2 QPV (\otimes^3 \Sigma) . \]

In fact, \( XX' \) is distributed as Wishart distribution.

**Lemma 3.1** Let \( P = P(1;1,1) \), \( Q = P(1,1;1) \). Then

(a) \( P^2 = I_{p^3} \quad Q^2 = I_{p^3} \).

(b) \( P^{-1} = P \), \( Q^{-1} = Q \).

(c) \( PQPQ = QP \), \( QPQP = PQ \).

**Proof.** (a) Using (1) and (2), we get \( Q^{-1} = Q \).

\[ P^2 = (I_p \otimes I_{p,p})(I_p \otimes I_{p,p}) = I_p \otimes I_{p^2} = I_{p^3}. \]

Similarly, we get \( Q^2 = I_{p^3} \).

(b) They are trivial because of (a) in lemma 3.1.

(c) Using (6), (7), and (8), we can derive

\[ PQPQ = (I_p \otimes I_{p,p})(I_{p,p} \otimes I_p)(I_p \otimes I_{p,p})(I_{p,p} \otimes I_p) \]
\[ = I_{p^2,p^2}I_{p^2,p^2} = I_{p^2,p^2} = QP \]

and

\[ QPQP = (I_{p,p} \otimes I_p)(I_p \otimes I_{p,p})(I_{p,p} \otimes I_p)(I_p \otimes I_{p,p}) = I_{p^2,p^2}I_{p^2,p^2} = I_{p,p^2} = PQ. \]
Lemma 3.2 \( QPQ (\otimes^3 \Sigma) = P (\otimes^3 \Sigma) QP \)

Proof. Let \( L = QPQ (\otimes^3 \Sigma) \). Then

\[
PLPQ = P (QPQ (\otimes^3 \Sigma)) PQ = QP (\otimes^3 \Sigma) PQ = \otimes^3 \Sigma
\]

by (5) and (c) in lemma 3.1.

Using \( P = P^{-1} \) and \( PQ = (PQ)^{-1} \), \( L = P (\otimes^3 \Sigma) QP \).

Lemma 3.3 \( QPQ (\otimes^3 \Sigma) = QP (\otimes^3 \Sigma) PQ = \otimes^3 \Sigma \)

Proof. They are trivial using (4) and (5).

Lemma 3.4 \( QPQ (\otimes^3 \Sigma) PQ = P (\otimes^3 \Sigma) \).

Proof. Let \( N = QPQ (\otimes^3 \Sigma) PQ \). Then

\[
PN = PQPQ (\otimes^3 \Sigma) PQ = QP (\otimes^3 \Sigma) PQ = \otimes^3 \Sigma
\]

by (5) and (c) in lemma 3.1.

Using \( P = P^{-1} \), \( N = P^{-1} (\otimes^3 \Sigma) = P (\otimes^3 \Sigma) \).

Theorem 3.5 Let \( a \times n \) random matrix \( X \) be distributed as \( N_{p,n} (0, \Sigma, \Phi) \).
Then \( E \left( S_X \otimes S_X \otimes ES_X \right) = n^2 V + n^3 (\otimes^3 \Sigma) + n^2 Q (\otimes^3 \Sigma) \).

Proof. Since \( ES_X = n \Sigma \) and \( E (\otimes^2 S_X) = n (\text{vec} \Sigma \text{vec}' \Sigma) + n^2 (\otimes^2 \Sigma) + n I_{p,p} (\otimes^2 \Sigma) \),

\[
E (S_X \otimes S_X \otimes ES_X) = E (\otimes^2 S_X) \otimes ES_X
= [n (\text{vec} \Sigma \text{vec}' \Sigma) + n^2 (\otimes^2 \Sigma) + n I_{p,p} (\otimes^2 \Sigma)] \otimes n \Sigma
= n^2 [(\text{vec} \Sigma \text{vec}' \Sigma) \otimes \Sigma] + n^3 (\otimes^3 \Sigma) + n^2 [I_{p,p} (\otimes^2 \Sigma) \otimes I_p \Sigma]
= n^2 V + n^3 (\otimes^3 \Sigma) + n^2 Q (\otimes^3 \Sigma)
\]

using (4).

Theorem 3.6 Let \( a \times n \) random matrix \( X \) be distributed as \( N_{p,n} (0, \Sigma, \Phi) \).
Then \( E \left( S_X \otimes ES_X \otimes S_X \right) = n^2 PQVQP + n^3 (\otimes^3 \Sigma) + n^2 P (\otimes^3 \Sigma) QP \).

Proof. Using (5), lemma 3.2, and lemma 3.3,

\[
E (S_X \otimes ES_X \otimes S_X) = E [I_{p,p^2} (S_X \otimes S_X \otimes ES_X) I_{p^2,p}]
= I_{p,p^2} E (S_X \otimes S_X \otimes ES_X) I_{p^2,p} = PQE (S_X \otimes S_X \otimes ES_X) QP
= PQ [n^2 V + n^3 (\otimes^3 \Sigma) + n^2 Q (\otimes^3 \Sigma)] QP
= n^2 PQVQP + n^3 PQ (\otimes^3 \Sigma) QP + n^2 P (\otimes^3 \Sigma) QP
= n^2 PQVQP + n^3 (\otimes^3 \Sigma) + n^2 P (\otimes^3 \Sigma) QP
\]

Theorem 3.7 Let \( a \times n \) random matrix \( X \) be distributed as \( N_{p,n} (0, \Sigma, \Phi) \).
Then \( E \left( ES_X \otimes S_X \otimes S_X \right) = n^2 QPVQP + n^3 (\otimes^3 \Sigma) + n^2 P (\otimes^3 \Sigma) \).
Proof. Using (5), lemma 3.3, and lemma 3.4,
\[
E(S_X \otimes ES_X \otimes S_X) = E[I_{p^2} (S_X \otimes S_X \otimes ES_X) I_{p^2}]
\]
\[
= QPE (S_X \otimes S_X \otimes ES_X) PQ
\]
\[
= QP [n^2V + n^3 (\otimes^3 \Sigma) + n^2Q (\otimes^3 \Sigma)] PQ
\]
\[
= n^2QPV PQ + n^3QP (\otimes^3 \Sigma) PQ + n^2QPQ (\otimes^3 \Sigma) PQ
\]
\[
= n^2QPV PQ + n^3 (\otimes^3 \Sigma) + n^2P (\otimes^3 \Sigma).
\]

Here is the key result of this paper.

**Theorem 3.8** \(E[\otimes^3 (S_X - ES_X)]\)
\[
= n [PV + VP + QPV + VPQ + PVQ + PQVP + PQ (\otimes^3 \Sigma)]
\]
\[
+ n^2 [PVP - PQVQP + QP (\otimes^3 \Sigma)] - 2n^3 (\otimes^3 \Sigma).
\]

Proof.
\[
E[\otimes^3 (S_X - ES_X)]
\]
\[
= E[(\otimes^3 S_X) + (ES_X \otimes S_X \otimes ES_X) + (ES_X \otimes ES_X \otimes S_X) + (S_X \otimes ES_X \otimes ES_X)]
\]
\[
- E[(ES_X \otimes S_X \otimes S_X) + (S_X \otimes S_X \otimes ES_X) + (S_X \otimes ES_X \otimes S_X) + (\otimes^3 ES_X)]
\]
\[
= E(\otimes^3 S_X) - E(S_X \otimes S_X \otimes ES_X) - E(S_X \otimes ES_X \otimes S_X)
\]
\[
- E(ES_X \otimes S_X \otimes S_X) + (\otimes^3 ES_X).
\]

Using theorem 3.6, theorem 3.7, and theorem 3.8,
\[
E[\otimes^3 (S_X - ES_X)]
\]
\[
= n [PV + VP + QPV + VPQ + PVQ + PQVP + PQ (\otimes^3 \Sigma)]
\]
\[
+ n^2 [PVP - PQVQP + QP (\otimes^3 \Sigma)] - 2n^3 (\otimes^3 \Sigma).
\]

**References**


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