Distribution of the Zeros of 
$q$-Stirling-Like Polynomials

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Abstract

In this paper, we observe an interesting phenomenon of ‘scattering’ of the zeros of the $q$-Stirling-Like polynomials $SR_{n,q}(x)$ in complex plane.

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1 Introduction

Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, Tangent numbers and polynomials, and Stirling numbers and polynomials possess many interesting properties and arising in many areas of mathematics, mathematical physics and statistical physics. It is the aim of this paper to investigate the roots of the $q$-Stirling-Like polynomials $SR_{n,q}(x)$. We also carried out computer experiments for doing demonstrate a remarkably regular structure of the complex roots of the $q$-Stirling-Like polynomials $S_{n,q}(x)$.

Throughout this paper, we always make use of the following notations: $\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the set of natural numbers, $\mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$ denotes the set of nonnegative integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers.
For a real or complex parameter \( x \), the Stirling polynomials are defined by the following generating function

\[
\sum_{n=0}^{\infty} S_n(x) \frac{t^n}{n!} = \left( \frac{t}{1 - e^{-t}} \right)^{x+1} \quad (|t| < 2\pi; 1^x := 1), \text{ see [1].}
\]

The Stirling-Like polynomials \( SR_n(x) \) and Stirling-Like numbers \( SR_n \) are defined by the following generating functions

\[
\sum_{n=0}^{\infty} SR_n(x) \frac{t^n}{n!} = \left( \frac{t}{1 - e^{-t}} \right) e^{xt}, \quad |t| < 2\pi, \tag{1.1}
\]

\[
\sum_{n=0}^{\infty} SR_n \frac{t^n}{n!} = \frac{t}{1 - e^{-t}}, \quad |t| < 2\pi, \tag{1.2}
\]

respectively.

In [2], we introduced the \((h, q)\)-Bernoulli numbers and polynomials. The polynomials \( B^{(h)}_{n,q}(x) \) and numbers \( B^{(h)}_{n,q} \) are defined by the following generating functions

\[
\sum_{n=0}^{\infty} B^{(h)}_{n,q}(x) \frac{t^n}{n!} = \left( \frac{h \log q + t}{q^h e^t - 1} \right) e^{xt}, \quad |\log q + t| < \pi, \tag{1.3}
\]

\[
\sum_{n=0}^{\infty} B^{(h)}_{n,q} \frac{t^n}{n!} = \frac{h \log q + t}{q^h e^t - 1}, \quad |\log q + t| < \pi, \tag{1.4}
\]

respectively. In the special case, \( h = 1 \), \( B_{n,q}(x) \) and \( B_{n,q} \) are called the \( n \)th \( q \)-Bernoulli polynomials and numbers, respectively. The \( q \)-Stirling-Like polynomials \( SR_{n,q}(x) \) and \( q \)-Stirling-Like numbers \( SR_{n,q} \) are defined by the following generating functions

\[
\sum_{n=0}^{\infty} SR_{n,q}(x) \frac{t^n}{n!} = \left( \frac{\log q - t}{q^e e^t - 1} \right) e^{xt}, \quad |\log q - t| < \pi, \tag{1.5}
\]

\[
\sum_{n=0}^{\infty} SR_{n,q} \frac{t^n}{n!} = \frac{\log q - t}{q^e e^t - 1}, \quad |\log q - t| < \pi, \tag{1.6}
\]

respectively. Note that \( SR_{n,q}(0) = SR_{n,q} \) and \( \lim_{q \to 1} SR_{n,q} = SR_{n,q} \). Clearly, we obtain

\[
SR_{n,q}(-x) = (-1)^n B_{n,q}(x) \quad \text{and} \quad SR_{n,q} = (-1)^n B_{n,q} \quad \text{for} \quad n \in \mathbb{N}_0,
\]
in terms of the $q$-Bernoulli polynomials $B_{n,n}(x)$ and Bernoulli numbers $B_{n,q}$.

By using computer, the $q$-Stirling-Like numbers $SR_{n,q}$ can be determined explicitly. Here is the list of the first $SR_{n,q}$’s numbers.

\[
SR_{1,q} = \log q - 1 + q,
SR_{2,q} = -1 - 1 + q + q \log q \left(-1 + q\right)^2,
SR_{3,q} = -2 q \left(-1 + q\right)^2 - q \log q \left(-1 + q\right)^2 + 2 q^2 \log q \left(-1 + q\right)^3.
\]

By the above definition, we obtain

\[
\sum_{l=0}^{\infty} BR_{l,q}(x) \frac{t^l}{l!} = \left(\log -t \over qe^{-t} - 1\right) e^t = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} SR_{n,q} \frac{t^n}{n!} \frac{t^m}{m!} x^n t^m
= \sum_{l=0}^{\infty} \sum_{n=0}^{l} SR_{n,q} \frac{t^n}{n!} x^{l-n} \frac{t^{l-n}}{(l-n)!} = \sum_{l=0}^{\infty} \sum_{n=0}^{l} \left(\frac{l}{n} SR_{n,q} x^{l-n}\right) \frac{t^l}{l!}.
\]

By using comparing coefficients $\frac{t^l}{l!}$, we have the following theorem.

**Theorem 1.1** For $n \in \mathbb{N}_0$, one has

\[
SR_{n,q}(x) = \sum_{l=0}^{n} \left(\frac{n}{l}\right) SR_{l,q} x^{n-l}.
\]

By Theorem 1.1, after some elementary calculations, we have

\[
\int_a^b SR_{n,q}(x) dx = \sum_{l=0}^{n} \left(\frac{n}{l}\right) SR_{l,q} \int_a^b x^{n-l} dx
= \sum_{l=0}^{n} \left(\frac{n}{l}\right) SR_{l,q} \frac{x^{n-l+1}}{n-l+1} \bigg|_a^b
= \frac{1}{n+1} \sum_{l=0}^{n+1} \left(\frac{n+1}{l}\right) SR_{l,q} x^{n-l+1} \bigg|_a^b.
\]

Hence we get

\[
\int_a^b SR_{n,q}(x) dx = \frac{SR_{n+1,q}(b) - SR_{n+1,q}(a)}{n+1}.
\]  

(1.3)

Since $SR_{n,q}(0) = SR_{n,q}$, by (1.3), we have the following theorem.
Theorem 1.2  For $n \in \mathbb{N}$, one has

$$SR_{n,q}(x) = SR_{n,q} + n \int_0^x SR_{n-1,q}(t)dt.$$  

Then, it is easy to deduce that $SR_{n,q}(x)$ are polynomials of degree $n$. By using computer, the polynomials $SR_{n,q}(x)$ can be determined explicitly. Here is the list of the first $q$-Stirling-Like $SR_{n,q}(x)$'s polynomials.

$$SR_{1,q}(x) = \frac{\log q}{-1 + q},$$

$$SR_{2,q}(x) = -\frac{1}{-1 + q} + \frac{q \log q}{(-1 + q)^2} + \frac{x \log q}{-1 + q},$$

$$SR_{3,q}(x) = -\frac{2q}{(-1 + q)^2} - \frac{q \log q}{(-1 + q)^2} + \frac{2q^2 \log q}{(-1 + q)^3} + \frac{2qx \log q}{(-1 + q)^2} + \frac{x^2 \log q}{-1 + q}.$$  

2 The zeros of the polynomials $SR_{n,q}(x)$

In this section, an interesting phenomenon of scattering of zeros of $SR_{n,q}(x)$ is observed. We investigate the beautiful zeros of the $SR_{n,q}(x)$ by using a computer. We plot the zeros of $SR_{n,q}(x)$ for $x \in \mathbb{C}$ (Figure 1). In Figure 1(top-left), we choose $n = 30$ and $q = 1/10$. In Figure 1(top-right), we choose $n = 30$ and $q = 3/10$. In Figure 1(bottom-left), we choose $n = 30$ and $q = 7/10$. In Figure 1(bottom-right), we choose $n = 30$ and $q = 9/10$. Our numerical results for numbers of real and complex zeros of $SR_{n,q}(x)$ are displayed in Table 2. In Table 1, we choose $q = 1/2$.

<table>
<thead>
<tr>
<th>degree $n$</th>
<th>real zeros</th>
<th>complex zeros</th>
<th>degree $n$</th>
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<th>complex zeros</th>
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<td>10</td>
<td>14</td>
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</tbody>
</table>
Figure 1: Zeros of $SR_{n,q}(x)$

We calculated an approximate solution satisfying $SR_{n,q}(x), x \in \mathbb{R}$. The results are given in Table 2. In Table 2, we choose $q = 1/2$.

**Table 2.** Approximate solutions of $SR_{n,q}(x) = 0$

<table>
<thead>
<tr>
<th>degree $n$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>$-0.442695$</td>
</tr>
<tr>
<td>2</td>
<td>$-0.727948, -0.157443$</td>
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<tr>
<td>3</td>
<td>$-0.947787, -0.419864, 0.0395652$</td>
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<td>4</td>
<td>$-1.12231, -0.668324, -0.154244, 0.1741$</td>
</tr>
<tr>
<td>5</td>
<td>$-1.2571, -0.899936, -0.39125, 0.104278, 0.23053$</td>
</tr>
<tr>
<td>6</td>
<td>$-1.34417, -1.1276, -0.628491, -0.126344$</td>
</tr>
<tr>
<td>7</td>
<td>$-0.861399, -0.358906, 0.138252$</td>
</tr>
<tr>
<td>8</td>
<td>$-1.09254, -0.592665, -0.0925458, 0.362362$</td>
</tr>
<tr>
<td>9</td>
<td>$-1.32193, -0.825457, -0.324809, 0.174575, 0.486913$</td>
</tr>
</tbody>
</table>
Stacks of zeros of \( SR_{n,q}(x) \) for \( 1 \leq n \leq 30 \) forming a 3D structure are presented (Figure 2).

In Figure 2 (top-right), we draw \( y \) and \( z \) axes but no \( x \) axis in three dimensions. In Figure 2 (bottom-left), we draw \( x \) and \( y \) axes but no \( z \) axis in three dimensions. In Figure 2 (bottom-right), we draw \( x \) and \( z \) axes but no \( y \) axis in three dimensions. We made a series of the following conjectures:

**Conjecture 1.** Prove that \( SR_n(x), x \in \mathbb{C} \), has \( Im(x) = 0 \) reflection symmetry analytic complex functions.

**Conjecture 2.** Prove that \( SR_{n,q}(x) = 0 \) has \( n \) distinct solutions.

Since \( n \) is the degree of the polynomial \( SR_{n,q}(x) \), the number of real zeros \( R_{SR_{n,q}}(x) \) lying on the real plane \( Im(x) = 0 \) is then \( R_{SR_{n,q}}(x) = n - C_{SR_{n,q}}(x) \), where \( C_{SR_{n,q}}(x) \) denotes complex zeros. See Table 1 for tabulated values of \( R_{SR_{n,q}}(x) \) and \( C_{SR_{n,q}}(x) \). The data concerning the numerical verification of Conjecture 1 and Conjecture 2 are contained in Tables 1 and 2. See Table 2 for tabulated values of \( R_{SR_{n,q}}(x) \) and \( C_{SR_{n,q}}(x) \). The theoretical prediction on the zeros of \( SR_{n,q}(x) \) is await for further study. For more studies and results in this subject, you may see [3, 4, 5, 6, 7, 8].

**References**


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