A Concept of Limiting Phase Trajectories and
Description of Highly Non-stationary
Resonance Processes

Leonid I. Manevitch

Semenov Institute of Chemical Physics of Russian Academy of Science
Kosygina, 4, Moscow, 119991, Russia

Abstract

We discuss a recently developed concept of limiting phase trajectories (LPTs) allowing a unified description of resonance, highly non-stationary processes for a wide range of classical and quantum dynamical systems with constant and varying parameters. This concept provides a far going extension and adequate mathematical description of the well-known linear beating phenomenon to a diverse variety of nonlinear systems ranging from classical multi-particle models up to nonlinear quantum tunneling. Contrary to stationary and non-stationary, but non-resonance oscillations, described in the framework of nonlinear normal modes (NNMs) concept, the non-stationary processes under consideration are characterized by strong modulation and intense energy exchange between different parts of the system. They include, e.g. targeted energy transfer, non-stationary vibrations of carbon nanotubes, quantum tunneling, auto-resonance and non-conventional synchronization. Besides the non-linear beating, the LPT concept allows to find the conditions of transition from intense energy exchange to strongly localized (e.g. breather-like) excitations. A special mathematical technique based on the non-smooth temporal transformations leads to the clear and simple description of strongly modulated regimes. The role of LPTs in the theory of resonance non-stationary processes turns out to be similar to that of NNMs in stationary case.

Keywords: Nonlinear normal modes; Weakly coupled nonlinear oscillators; Energy transfer; Limiting phase trajectory
1 Introduction

The accepted classification of the problems of mathematical physics (in models of the oscillation and wave theory) draws first of all a sharp distinction between linear and nonlinear model [1-5]. Such a distinction is caused by understandable mathematical reasons including absence of superposition principle in the nonlinear case. However it was recently shown [6-13] that in-depth physical analysis allows us to introduce other basis for classification of the oscillation problems, focusing on the difference between stationary (or non-stationary, but non-resonance) and resonance non-stationary processes. In the latter case a discrimination of linear and nonlinear problems is not fundamental if we deal with regular (non-chaotic) motions, and a specific technique has been developed which is efficient in the same degree for description of both linear and nonlinear resonance non-stationary processes. The existence of alternative approach in the framework of linear theory seems unexpected. Really, the superposition principle allows us to find a solution describing arbitrary non-stationary oscillations as a combination of linear normal modes which correspond to stationary processes. However, in the systems of weakly coupled oscillators, in which resonance non-stationary vibrations can occur, other type of fundamental solution exists. It describes strongly modulated non-stationary oscillations characterized by the maximum possible energy exchange between the oscillators or the clusters of the oscillators (effective particles). Such solutions are referred to as Limiting Phase Trajectories (LPTs). It was demonstrated that the LPT concept suggests a unified approach to the study of highly non-stationary processes in a wide range of classical and quantum dynamical systems with constant and time-varying parameters [12].

The development and use of the analytical framework based on the LPT concept is motivated by the fact that resonance non-stationary processes occurring in a broad variety of finite dimensional physical models are beyond the well-known paradigm of nonlinear normal modes (NNMs), fully justified only for quasi-stationary and non-stationary, but non-resonance processes. While the NNMs approach has been proved to be an effective tool for the analysis of stationary regimes, their instability and bifurcations (see, e.g., [2, 3]), the use of the LPTs concept provides the adequate procedures for studying strongly non-stationary regimes as well as the transitions between different types of non-stationary motions, including propagation of localized excitations [7-8]. It makes possible, at the first time, to extend the notion of beating phenomenon to the systems with many degrees of freedom. Moreover, the concept of the limiting phase trajectories allows the prediction of the new type of synchronization (LPT-synchronization) in the system of weakly coupled autogenerators [9] and this is in contrast to the conventional NNM-synchronization [14]. Note that, along with the well-known asymptotic methods, the investigation of the phenomena under consideration has required the application of the special technique of non-smooth temporal transformations providing a simple description of strongly
modulated and transient regimes. This technique was initially elaborated for description of vibro-impact (or close to them) processes [15].

The paper contains a brief review of several significant applications of the LPT concept. To illustrate its main constituents we use the model of two weakly coupled identical nonlinear oscillators (Section 2). We demonstrate also that LPT can be an attractor considering two weakly coupled autogenerators with hard excitation (Section 3). In the last section we discuss an extension of the concept to many degrees of freedom systems.

We do not stop here on the applications to quantum mechanics [12], nano-scale structures [14] and to forced oscillations [10].

2 Two weakly coupled nonlinear oscillators

Beats between two weakly coupled classical linear oscillators represent the simplest example of resonance non-stationary process consisting in complete energy exchange between the parts of the system. In quantum mechanics, beats correspond to periodic oscillations of the probabilities for finding the system in one of the two basis states (in the two-level model). Due to the superposition principle, linear beats can be described as a combination of the normal modes corresponding to the stationary states in the slow time. But this representation is not applicable for nonlinear oscillators wherein the superposition principle does not hold. The absence of an adequate analytical description of nonlinear strongly modulated processes prevented the understanding of their specifics and restricted the analysis by the numerical results, which do not allow further theoretical generalizations. For example, until recently there was no extension of beat phenomenon to many-body systems. We show that LPT presents an adequate description of both linear and nonlinear beats in the systems with two and many degrees of freedom. Moreover, it turns out that in the system of weakly coupled active oscillators LPT can be the attractor that means existence of non-conventional LPT-synchronization which is an alternative to conventional NNM-synchronization.

As for the case of forced vibrations, LPT concept presents a specific key to study of strongly modulated vibrations corresponding to maximally possible, under given conditions, energy inflow from its source to the oscillating system. In spite of the fact that LPTs are always determined by nonlinear equations, the elaborated technique provides their efficient solution in terms of non-smooth basic functions. The only qualitative difference between linear and nonlinear (in conventional sense) problems is transition to energy localization which is possible in nonlinear case only.

Let us consider the system of two weakly coupled nonlinear oscillators (Fig.1).
Furthermore, combinations of non-resonant NNMs in quasi-linear systems can be used for an asymptotic description of non-stationary processes in both finite and infinite models. In these approaches, a difference between the linear and nonlinear cases consists only in the amplitude dependence of the natural frequencies of nonlinear systems. As for energy transfer and changes in the state of system in the infinite models, they can be described by solitonic excitations.

However, non-stationary energy transfer and exchange, as well as the change in the system state under the condition of resonance in a finite oscillatory system are characterized by complicated non-stationary behaviour, for which any analytical investigation is prohibitively difficult. In this paper we discuss a new concept, which provides a unified description of highly non-stationary resonance processes for a wide range of classical and quantum dynamical systems with constant and time-variant parameters.

To this end, we introduce the notion of the Limiting Phase Trajectory (LPT) and show that the role of the LPTs in a deeper understanding and the description of resonant highly non-stationary processes is similar to the role of the NNMs in the analysis of stationary and non-stationary non-resonance regimes.

Hamilton function of this system (in dimensionless variables) has a view:

\[ H = \sum_{j=1,2} \left( \frac{1}{2} \left( \frac{dU_j}{d\tau_j} \right)^2 + \frac{1}{2} U_j^2 + 2\varepsilon\alpha U_j^4 \right) + \frac{1}{2} \varepsilon\beta(U_1 - U_2)^2, \]

(2.1)

where \( U_j = \frac{u_j}{l_0} \), \( \tau_j = \sqrt{\frac{c_j}{m}} t \), \( 8\alpha\varepsilon = \frac{c_j l_0^2}{c_1} \), \( 2\varepsilon\beta = \frac{c_{12}}{c_1} \), \( \varepsilon << 1 \), \( m \) – mass of single oscillator, \( l_0 \) – length of every unreformed spring. According to the assumption \( \varepsilon << 1 \) not only the coupling is weak but also the nonlinearity has to be small.

Corresponding equations of motion can be written as follows:

\[ \frac{d^2U_1}{d\tau_1^2} + U_1 + 2\varepsilon\beta(U_1 - U_2) + 8\varepsilon\alpha U_1^3 = 0, \]

\[ \frac{d^2U_2}{d\tau_1^2} + U_2 + 2\varepsilon\beta(U_2 - U_1) + 8\varepsilon\alpha U_2^3 = 0. \]

(2.2)

Using complex representation of the equations of motion with respect to variables

![Schematic representation of the model](image-url)
A concept of limiting phase trajectories

\[ \varphi_j = \left( \frac{d}{d \tau} U_j + i U_j \right) e^{i \tau_j} \]  
(2.3)

and two scale expansions

\[ \varphi_j(\tau_1, \tau_2) = \sum_{n=0} \varphi_{j,n}(\tau_1, \tau_2) e^n \]  
(2.4)

one can obtain in main asymptotic approach the equations of motion in slow time

\[ \frac{\partial f_1}{\partial \tau_2} + i \beta f_2 - 3i \alpha |f_1|^2 f_1 = 0 \]  
(2.5)

\[ \frac{\partial f_2}{\partial \tau_2} + i \beta f_1 - 3i \alpha |f_2|^2 f_2 = 0. \]

where \( \varphi_{j,0} = e^{i \beta \tau_2} f_j, \quad j = 1, 2; \quad \tau_2 = \epsilon \tau_1, \quad \frac{d}{d \tau_1} \to \frac{\partial}{\partial \tau_1} + \epsilon \frac{\partial}{\partial \tau_2} \)

with two integrals of motion

\[ H_1 = \beta (f_2 f_1^* + f_1 f_2^*) - \frac{3}{2} \alpha (|f_1|^4 + |f_2|^4) \]
\[ N = |f_1|^2 + |f_2|^2. \]  
(2.6)

Transition to real variables \( f_1 = \sqrt{N} \cos \theta \ e^{i \delta_1}, \ f_2 = \sqrt{N} \sin \theta \ e^{i \delta_2} \) leads to following presentations of Hamilton function and equations of motion

\[ H_1 = N(\cos \Delta + k \sin 2\theta) \sin 2\theta \]
\[ \frac{d\theta}{d \tau_2} = \beta \sin \Delta, \quad \sin 2\theta \frac{d\Delta}{d \tau_2} = 2 \beta \cos 2\theta \cos \Delta + \frac{3}{2} \alpha N \sin 4\theta. \]  
(2.7)

where \( \Delta = \delta_1 - \delta_2, \ k = \frac{3\alpha N}{4\beta}, \ \alpha > 0. \)

This system (2.7) is strongly nonlinear even in the case of initially linear problem.

Let us present plots of the phase trajectories for different values of \( k \) in Fig.2.

Stationary points correspond to NNMs (stationary processes) and LPT corresponds to maximum possible energy exchange between oscillators which is resonance highly nonstationary process. The evolution of out-of-phase NNM demonstrates its well-known instability at \( k = 0.5 \) and birth of the new two stable NNMs with separatrix encircling them. However, there is once more qualitative change of the phase plane caused by coincidence of LPT and separatrix, leading
to disappearance of complete energy exchange between oscillators and predominant energy localization on one of them when \( k > 1 \) (Fig. 2f).

![Phase plane: evolution of NNMs and LPT with growth of \( k \).](image)

We present also in Fig. 3 an alternative demonstration of the LPT evolution on the phase plane for equation

\[
\ddot{\theta} + k^2 \sin 4\theta = 0, \quad 0 \leq \theta \leq \frac{\pi}{2}
\]  

(2.8)

(where \( \dot{} \) – is derivative with respect slow time \( \tau_2 \)) which is valid for LPTs only. Because of the restriction for \( \theta \), the LPTs which are situated far out of separatrix correspond to almost linear beats. When approaching to separatrix strongly nonlinear beats with complete energy exchange appear. As for the LPTs inside the separatrix, they reflect predominant energy localization after coincidence of LPT with separatrix.
A concept of limiting phase trajectories

Fig. 3. Phase plane $\theta, \dot{\theta}$ for LPT.

Exact solution for LPT in terms of $\theta, \Delta$ is presented graphically in Fig. 4 for $k = 0.5$ [16]. It resembles a saw tooth function and its derivative in sense of distributions theory (Fig. 5). Therefore non-smooth transformations with introducing the functions $\tau, e$ [17] unexpectedly turn out to be adequate mathematical tool for study of highly non-stationary resonance processes.

Fig. 4. The exact solution for $k = 0.5$

The advantages of the techniques based on the use of the non-smooth variables are evident while dealing with the nonlinear beats. They can not be presented as linear combination of NNM because the superposition principle is not valid in this case. It was shown earlier [6, 8, 18–22], that an efficient temporal description of LPT in nonlinear chains is attained in the terms of the non-smooth functions of slow time $\tau(\tau_2), e(\tau_2)$ (Fig. 2). Then the dependent variables can be presented as [17]
\[ \theta = X_1(\tau) + Y_1(\tau) e \left( \frac{\tau_2}{a} \right), \quad \Delta = X_2(\tau) + Y_2(\tau) e \left( \frac{\tau_2}{a} \right) \]  \hspace{1cm} (2.9)

The possibility of similar substitutions is based on the statement that every periodic process, independently on the class of its smoothness, is expressed by the unique manner as an element of the algebra of hyperbolic numbers through the variables \( \tau \) and \( e \) \[17\]:

\[ x(\tau_2) = x(\tau, e) = X(\tau) + eY(\tau), \quad e(\tau_2) = \frac{d\tau}{d\tau_2}. \] \hspace{1cm} (2.10)

where \( X(\tau) = \frac{1}{2}[x(\tau) + x(2 - \tau)] \), \( Y(\tau) = \frac{1}{2}[x(\tau) - x(2 - \tau)] \)

so, that \( x(\tau, e) \equiv x(\tau) + eY(\tau(t_2), e(t_2)) \).

At that, the pair \( (1, e) \), where \( e^2 = 1 \) is a basis, and the algebraic operations as well as differentiation or integration over time preserve the structure of hyperbolic number. This property provides applicability and convenience of the corresponding transformations while solving the differential equations \[17\].

Interestingly that the hyperbolic numbers which are frequently used for a simplest illustration of the Clifford algebra, were known from the middle of XIX century as abstract mathematical objects without any connection with vibration processes. On the other side, the elliptic complex numbers with the basis \( \{1, i\} \) \((i^2 = -1)\) and corresponding trigonometric functions turned out, in essence, the main tool for the description of such processes.

Using the non-smooth transformations we can present solution of Eqs. 2.7 as

\[ \theta = X_1(\tau) + Y_1(\tau) e \left( \frac{\tau_2}{a} \right), \quad \Delta = X_2(\tau) + Y_2(\tau) e \left( \frac{\tau_2}{a} \right) \]

Substitution of expression (2.9) into equations (2.7) leads to “smooth equations with respect to non-smooth variables”:

\[ \frac{\partial}{\partial \tau} \left\{ \begin{array}{c} X_1 \\ Y_1 \end{array} \right\} = \frac{1}{2} a\beta [\sin(X_2 + Y_2) \mp \sin(X_2 - Y_2)] \]

\[ \frac{\partial}{\partial \tau} \left\{ \begin{array}{c} X_2 \\ Y_2 \end{array} \right\} = a\beta [\sec \theta (X_1 + Y_1) \cos(X_2 + Y_2) \mp \sec \theta (X_1 - Y_1) \cos(X_2 + Y_2)] \]

\[ + \frac{3a}{2} \alpha \beta [\cos \theta (X_1 + Y_1) \mp \cos \theta (X_1 - Y_1)]. \] \hspace{1cm} (2.11)

Their solution in power series is presented as follows:
For linear beat we have really exact one-term saw-tooth function and its derivative

\[ X_{1,0} = 0, \quad X_{1,1} = \frac{\pi}{2}, \quad Y_{1,0} = 0, \quad Y_{2,0} = \frac{\pi}{2}. \]  
(2.12)

For nonlinear beat one can find some corrections for both profiles

\[ X_{1,0} = 0, \quad X_{1,1} = \alpha \beta, \quad X_{1,3} = \frac{2}{3} (\alpha \beta)^3 \overline{k^2}, \]
\[ Y_{2,1} = 2 \alpha \beta k, \quad Y_{2,3} = \frac{2}{3} (\alpha \beta)^3 \overline{k}, \]  
(2.13)

but periodicity is already taken into account due to non-smooth substitution.

3 Two weakly coupled autogenerators

The natural question is: can LPT be the attractor? To answer on this question we have considered [9] the system of weakly coupled active oscillators with hard excitation. Corresponding equations of motion in dimensionless form are written as follows:

\[ \frac{d^2 u_1}{dt^2} + u_1 + 8 \alpha \varepsilon u_1^3 + 2 \beta \varepsilon (u_1 - u_2) + 2 \varepsilon (\gamma - 4 \beta u_1^2 + 8 \varepsilon u_1^4) \frac{du_1}{dt} = 0; \]
\[ \frac{d^2 u_2}{dt^2} + u_2 + 8 \alpha \varepsilon u_2^3 + 2 \beta \varepsilon (u_2 - u_1) + 2 \varepsilon (\gamma - 4 \beta u_2^2 + 8 \varepsilon u_2^4) \frac{du_2}{dt} = 0. \]  
(3.1)

Small parameter \( \varepsilon \) (\( \varepsilon << 1 \)) characterizes weak coupling, nonlinearity and dissipation. Using the procedure similar to that presented above for conservative system one can derive the equations for the same variables \( \theta \) and \( \Delta \) characterizing the relationship between amplitudes of active oscillators and phase shift between them. In contrast to conservative system, reduction to such variables is possible only under additional conditions which have been revealed with using Lie group theory:

\[ b^2 = 9 \gamma \frac{d}{2}, \quad N = \frac{2b}{3d}. \]  
(3.2)

These conditions provide the symmetry of the equations of motion, allowing existence of the first integral \( N = R_i^2 + \overline{R_i}^2 \) (where \( R_i \) is modulus of complex functions \( \varphi_i \)) in spite of the fact that the system under consideration is not conservative one.
Fig. 6. Evolution of phase plane. Intensive energy exchange

$$0 < \lambda < 0.5 \left( 1 + \sqrt{1 - 4k^2} \right).$$

Energy localization

$$\lambda \geq 0.5 \left( 1 + \sqrt{1 - 4k^2} \right).$$

Evolution of the phase plane with change of the system parameters is presented in Fig. 6. The stationary points correspond to unstable limiting circles with energy distribution between oscillators which is determined by in-phase and out-of-phase NNMs. Limiting phase trajectories describe non-conventional LPT-synchronization which is an alternative to conventional NNM-synchronization. The change of the type of stationary states does not lead to qualitative change of LPT. Only its instability leads to appearance of localized solutions corresponding to limiting cycles with predominant energy concentration in one of oscillators. Thus, the LPT really can be the attractor and its knowledge allows predicting a transition to energy localization. Moreover, similarly to the case of conservative system one can find its analytical representation in terms of non-smooth basic functions $\tau, e$ which is plotted in Fig. 6:

$$\theta = \frac{\pi}{2} \tau + e \left( -\frac{\pi \lambda}{2} \tau^2 \right) + \ldots;$$

$$\Delta = e \left( \frac{\pi}{2} + k \tau \right) + \frac{-k\lambda}{3} \tau^2 + \ldots$$

(3.3)
4. Strongly Modulated Processes in Nonlinear Systems

Before development of the soliton theory, the only approach available for description of the energy transfer was built on the spreading (due to dispersion) wave packet concept. The theory of solitons gave birth to a new approach to the problem: it was shown that the dispersion effects can play the central role because they compensate the inevitable nonlinear distortion in the wave packet (or, vice versa, the nonlinearity compensates the inevitable in the linear dynamics dispersion "spreading" of the wave packet). The energy transfer in ordered systems can also be executed by the mobile "vibration solitons", or breathers. In this case, physical reasons for the mutual compensation of nonlinear and dispersive effects are hidden behind a powerful mathematical tool - the inverse scattering method. This method reveals a high degree of symmetry in certain classes of nonlinear systems, which enables the formation of the stable solitons [5].

In the system with many degrees of freedom an adequate understanding of the intense energy exchange and transition to energy localization is achieved with the introduction of "effective particles", whose dynamics is described by the LPTs [7, 23]. In essence, this approach clarifies also the physical nature of the breathers formed in the oscillatory chain with a large number of particles, in particular, in the polymer chains. At the same time, there is a range of the initial conditions, exposure to which determines the validity of NNMs concept. Let us dwell on the finite periodic Fermi-Pasta-Ulam (FPU) chain with Hamilton function

\[
H = \sum_{j} \frac{p_j^2}{2m} + V(Q_{j+1} - Q_j)
\]

\[
V(x) = \frac{1}{2}x^2 + \frac{\alpha}{3}x^3 + \frac{\beta}{4}x^4
\]

(4.1)

\[
Q_{N+1} = Q_1
\]

The number of particles in Eq. (5.1) assumed to be even. The generalization to the asymmetric system is given in [24, 25]. The strong coupling between the parti-
icles induces, as usually, the transition to the normal coordinates with using the canonical transformation.

The frequencies of the linearized system are defined by the relation
\[ \omega_k = 2 \sin \left( \frac{\pi k}{N} \right), \quad k = 0, \ldots, N - 1. \]
As in the case of oscillators, we introduce the complex variables and rewrite the dynamic equations in complex form:

\[
\begin{align*}
\Psi_k &= \frac{1}{\sqrt{2}} \left( \frac{d\zeta_k}{dt} + i \omega_k \zeta_k \right), \\
\Psi_k^* &= \frac{1}{\sqrt{2}} \left( \frac{d\zeta_k}{dt} - i \omega_k \zeta_k \right)
\end{align*}
\]

(4.2)

The equations of motion in the variables \((5.1)\) can be written as follows:

\[
\begin{align*}
i \frac{d\Psi_k}{dt} + \omega_k \Psi_k - \frac{\beta}{8N} \omega_k \sum_{l,m,n=1}^{N-1} C_{k,l,m,n} (\Psi_l - \Psi_l^*) (\Psi_m - \Psi_m^*) (\Psi_n - \Psi_n^*) &= 0
\end{align*}
\]

(4.3)

The original formulation of the problem does not contain a small parameter. However, as it was mentioned above, with the increase of the particles number a densification of the frequency spectrum in its upper part is observed. Mathematical reflection of this fact is the appearance of the quantity \(1/N\) in the formula for the natural frequencies, which will be considered as a small parameter appropriate for construction of the asymptotic expansion.

Presentation of the frequencies \(\omega_{N_k}/2\) in the form:

\[
\omega_{N_k}/2 = 2 \sin \left[ \frac{\pi}{N} \left( \frac{N}{2} \pm 1 \right) \right] = 2 \cos \left( \frac{\pi}{N} \right) \approx \omega_{N_k}/2 \left[ \frac{1 - \left( \frac{\pi}{N} \right)^2}{2} \right],
\]

(4.4)

where \(\omega_{N_k}/2\) - the upper limit frequency in the first Brillouin zone, confirms the spectrum densification at higher frequencies with increasing the number of the particles.

In accordance with the procedure of the two-scales expansions we supposed that

\[
\begin{align*}
\Psi_k &= \phi_k e^{i \nu_0 t}, \\
\phi_k &= \sqrt{\varepsilon} \left( \chi_{k,1} + \varepsilon \chi_{k,2} + \varepsilon^2 \chi_{k,3} + \ldots \right), \\
\tau_0 &= t; \tau_1 = \varepsilon t; \tau_2 = \varepsilon^2 t; \varepsilon = 1/N
\end{align*}
\]

(4.5)

We write the dynamic equations of the main asymptotic approximation for the highest-frequency mode and two closest in frequency modes of the spectrum (4.4) (it is clear that such a restriction is possible due to much weaker coupling with other modes)
\[ i \frac{d\chi_{N/2}}{d\tau} + \frac{3\beta}{4} \left( \chi_{N/2} \right)^2 \chi_{N/2} + 2 \left( \chi_{N/2-1} \right)^2 \chi_{N/2} + \left( \chi_{N/2-1}^2 + \chi_{N/2+1}^2 \right) \chi_{N/2} = 0 \]

\[ i \frac{d\chi_{N/2-1}}{d\tau} - \frac{\pi^2}{2} \chi_{N/2-1} + \frac{3\beta}{8} \left[ \left( \chi_{N/2-1} \right)^2 + 3 \left( \chi_{N/2-2} \right)^2 \right] \]

\[ \frac{d\chi_{N/2+1}}{d\tau} - \frac{\pi^2}{2} \chi_{N/2+1} + \frac{3\beta}{8} \left[ \left( \chi_{N/2+1} \right)^2 + 3 \left( \chi_{N/2+2} \right)^2 \right] \]

\[ X = \left( \left( \chi_{N/2} \right)^2 + \left( \chi_{N/2-1} \right)^2 + \left( \chi_{N/2-2} \right)^2 \right) = \text{const} \]

Here we omit the second indices of the functions corresponding to the main asymptotic approximation.

The transition from the waves to the effective particles is implemented through the change of variables

\[ \psi_1 = \frac{\chi_{N/2}}{\sqrt{2}} - \frac{\sqrt{1-2c^2}}{\sqrt{2}} \chi_{N/2-1} - c \chi_{N/2+1} \]

\[ \psi_2 = \frac{\chi_{N/2}}{\sqrt{2}} + \frac{\sqrt{1-2c^2}}{\sqrt{2}} \chi_{N/2-1} + c \chi_{N/2+1} \]

\[ \varphi = \sqrt{2c} \chi_{N/2-1} - \sqrt{1-2c^2} \chi_{N/2+1} \]

The account of the three (not two) modes associates with the degeneracy of the normal modes which are close in frequency to the upper boundary of the spectrum. Parameter “c” reflects the ratio of the degenerate modes contributions in the initial conditions and varies from zero to unit, but it is not presented in the equations of motion for the effective particles.

Preserving the invariant magnitude \( X = |\psi_1|^2 + |\psi_2|^2 + |\varphi|^2 \), we consider for definiteness the case \( \varphi = 0 \). Since this invariant quantity is the integral of motion in the slow time, \( \psi_1 \) and \( \psi_2 \) can be expressed in terms of the angular variables:

\[ \psi_1 = \sqrt{X} \cos \theta e^{i\phi}; \quad \psi_2 = \sqrt{X} \sin \theta e^{i\phi}. \]

Then the Hamiltonian in these variables takes the form:

\[ H(\theta, \Delta) = \frac{X}{64} \left[ 27\beta X - 16\pi^2 + 2(8\pi^2 - 3\beta X) \cos \Delta \sin 2\theta - 3\beta X (8 - \cos^2 \Delta) \sin^2 2\theta \right], \]
where $\Delta = \delta_2 - \delta_1$.

The equations of motion corresponding to Hamiltonian (5.9) are written as follows

$$
\frac{d\theta}{d\tau_2} + \frac{1}{32}[8\pi^2 - 3\beta X (1 - \cos \Delta \sin 2\theta)] \sin \Delta = 0
$$

$$
\sin 2\theta \frac{d\Delta}{d\tau_2} - \frac{1}{32} \cos 2\theta [(8\pi^2 - 3\beta X) \cos \Delta - 3\beta X (8 - \cos^2 \Delta \sin 2\theta)] = 0
$$

(4.10)

Let us consider the phase trajectory that characterizes the transition between the states $\psi_1 (\theta = 0)$ and $\psi_2 (\theta = \pi/2)$ has two branches and encloses the family of trajectories encircling the stationary points. This is the LPT as well as in the case of two weakly coupled oscillators.

The corresponding temporal process leads to complete energy exchange between parts of the chain which are denoted as effective particles. Presented dependences clearly demonstrate the adequacy of the concept of the effective particles and LPTs.

The graphs in Figs. 11-14 reflect the evolution of the phase plane with an increase in the nonlinearity parameter, which in this case is denoted as $\beta X$. They also illustrate the transition from the intense energy exchange (strongly modulated oscillations) to the energy localization. The first transition mentioned in the captions is associated with the dynamic instability of the boundary normal mode, followed by the birth of two new stable normal modes, and the separatrix separating them. The full energy exchange between the effective particles, which is described by LPT, still remains possible (Fig. 12). The second transition occurs at the coincidence of the separatrix and the LPT, which leads to the impossibility of the full energy exchange. As a result, one can observe dominant energy localization on the excited effective particle (Fig. 13). The formation of a mobile localized excitation [7, 24, 25] also becomes possible.
**Fig. 12.** The phase plane and energy exchange between the first and second dynamical transitions.
In the case corresponding to Fig. 11, a periodic FPU chain contains ten particles, from which the two effective particles are formed. The mutual energy exchange is described by LPT.

Thus, similarly to 2DoF systems, there are two energy thresholds here. They define respectively the instability of the boundary normal mode and merge of LPT with the separatrix which divides the domains of attraction of the new normal modes produced by the bifurcation of the boundary mode. However, the energy thresholds disappear, when the number of the particles as well as the resonances, goes to infinity; and thus it becomes possible an alternative, continuum approach. In this limit, one can use the inverse scattering method [26] in the long-wavelength and short-wavelength approximation (modified KdV and NSE, respectively). But then we cannot identify the stage of the intense energy exchange (strongly modulated vibrations) and the transition to the energy localization, which is specific only for the finite systems. Thus, the physical aspect of the formation of a localized excitation (a weak interaction between the effective particles) remained to be unclarified. Prior to the limiting transition this aspect is crucial, and the concept of the effective particle and the LPT can describe analytically the intense energy exchange and localization in the finite chains of weakly interacting oscillators.

The analytical presentation of the solution for the FPU chain in the terms of power series over slow time (the periodicity of the process is taken into account by introducing the independent variable \( \tau \)) (4.11)
A concept of limiting phase trajectories

\[ X_i = \sum_{l=0}^{\infty} X_{j, l} \tau^l, \quad Y_i = \sum_{l=0}^{\infty} Y_{j, l} \tau^l \]  \hspace{1cm} (4.11)

where \( j=1,2 \), is illustrated in Fig.15.

**Fig 15.** Time behavior of the symmetric FPU chain corresponding to LPT in coordinates \( \theta, \Delta \).

Close results can be obtained for the asymmetric FPU potential [25]. The extension to the case of the chain interacting with an elastic foundation is also possible. Then a minimum frequency in the spectrum of the linearized system differs from zero [23]. The effective particles in this case are formed, contrary to the FPU chain, in the low part of the spectrum (closely to the frequency gap) if one deals with a soft nonlinearity (similarly to the Frenkel-Kontorova model). With such change all the results presented above are valid. They can be applied to the analysis of the nonlinear dynamics of crystalline oligomers in the vicinity of the optic branch of the dispersion curve. This allows one to clarify the most efficient mechanisms of the energy exchange and the transition to energy localization in such systems. Let us note that the localization of the vibrations in polyatomic molecules and their role in the relaxation processes are considered in monograph [27] (see also citations there).
5 Conclusion

Adequate analysis of strongly modulated processes in nonlinear dynamics goes out of framework of the existing paradigm. The concept of Limiting Phase Trajectories which turns out to be an alternative to Nonlinear Normal Modes concept gives an efficient tool for such analysis. The mathematical content of this concept is closely connected with non-smooth transformations which were used earlier for study of vibro-impact processes. Finally, we present a comparison of two basic concepts of finite dimensional nonlinear dynamics.

<table>
<thead>
<tr>
<th>Nonlinear Normal Mode</th>
<th>Limiting Phase Trajectories</th>
</tr>
</thead>
<tbody>
<tr>
<td>- elementary stationary process</td>
<td>- elementary strongly non-stationary process</td>
</tr>
<tr>
<td>- is not involved into processes of energy exchange</td>
<td>- describes maximum possible (under given conditions) energy exchange between different parts of the system</td>
</tr>
<tr>
<td>- can be localized (localized NNM stationary localization)</td>
<td>- can be localized (localized LPT non-stationary localization)</td>
</tr>
<tr>
<td>- can bifurcate (transformation to localized NNMs)</td>
<td>- can bifurcate (transformation to localized LPT)</td>
</tr>
<tr>
<td>- can be attractor in active system</td>
<td>- can be attractor in active system</td>
</tr>
<tr>
<td>- in the presence of forcing is transformed into steady-state oscillations</td>
<td>- in the presence of forcing is transformed into LPT describing maximum possible taking away from the energy source</td>
</tr>
<tr>
<td>- can be presented by a sine-like basic functions</td>
<td>- can be presented by non-smooth basic functions</td>
</tr>
</tbody>
</table>
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References


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