A Note on Three Variable Symmetric Identities for Modified $q$-Bernoulli Polynomials
Arising from Bosonic $p$-Adic Integral on $\mathbb{Z}_p$

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Abstract

Recently, Dolgy-Kim-Kim derived some identities of Carlitz’s $q$-Bernoulli polynomials under symmetry group $S_3([5])$. In this paper, we investigate identities of
symmetry for the modified Carlitz’s $q$-Bernoulli polynomials which are different the symmetric identities of Dolgy-Kim-Kim for the Carlitz’s $q$-Bernoulli polynomials.

**Keywords:** three variable symmetric identities for modified $q$-Bernoulli polynomials

1. Introduction

Let $p$ be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of $p$-adic integers, the field of $p$-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The $p$-adic norm is normally defined by $|p|_p = \frac{1}{p}$. Let $q$ be an indeterminate in $\mathbb{C}_p$ such that $|1 - q|_p < p^{-1/p-1}$ and let the $q$-extension of number $x$ be defined by $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the $p$-adic $q$-integral on $\mathbb{Z}_p$ is defined by Kim to be

\[
I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x)\mu_q(x + p^N\mathbb{Z}_p) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x, \quad \text{(see [8]).}
\]

From (1.1), we note that

\[
q \int_{\mathbb{Z}_p} q^{-x}f(x+1)d\mu_q(x) - \int_{\mathbb{Z}_p} q^{-x}f(x)d\mu_q(x) = \frac{q-1}{\log q} f'(0).
\]

By (1.2), we easily get

\[
q^n \int_{\mathbb{Z}_p} q^{-x-n}f(x+n)d\mu_q(x) - \int_{\mathbb{Z}_p} q^{-x}f(x)d\mu_q(x) = \frac{q-1}{\log q} \sum_{l=0}^{n-1} f'(l).
\]

The Carlitz’s $q$-Bernoulli numbers are defined as

\[
\beta_0,q = 1, \ q(q\beta_q + 1)^n - \beta_n,q = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases}
\]

with the usual convention about replacing $\beta^n_q$ by $\beta_{n,q}$ (see [2], [3]). In [8] and [9], Kim gave the integral representation of Carlitz’s $q$-Bernoulli numbers on $\mathbb{Z}_p$ as follows:
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$$\beta_{n,q} = \int_{Z_p} [x]^n d\mu_q(x), \ (n \geq 0). \quad (1.5)$$

The Carlitz’s $q$-Bernoulli polynomials are given by

$$\beta_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} q^lx \beta_{l,q}[x]^{n-l}, \ (n \geq 0). \quad (1.6)$$

From (1.5) and (1.6), we have

$$\int_{Z_p} [x+y]^n d\mu_q(y) = \beta_{n,q}(x), \ (n \geq 0), \ \text{(see [1] – [11])}. \quad (1.7)$$

Recently, Dolgy-Kim-Kim gave some interesting identities of symmetry for the Carlitz’s $q$-Bernoulli polynomials (see [5]). In this paper, we consider the modified Carlitz’s $q$-Bernoulli numbers as follows:

$$B_{0,q} = \frac{q - 1}{\log q}, (qB_q + 1)^n - B_{n,q} = \delta_{1,n}, \ \text{(see [11])}, \quad (1.8)$$

with the usual convention about replacing $B^n_q$ by $B_{n,q}$.

From (1.2), we note that

$$B_{n,q} = \int_{Z_p} q^{-x} [x]^n d\mu_q(x), \ (n \geq 0). \quad (1.9)$$

The modified $q$-Bernoulli polynomials are given by

$$B_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} q^lx B_{l,q}[x]^{n-l} \quad (1.10)$$

$$= \int_{Z_p} [x+y]^n q^{-y} d\mu_q(y), \ \text{(see [11])}.$$  

The purpose of this paper is to give some new identities of the symmetry for the modified Carlitz’s $q$-Bernoulli polynomials which are different the symmetric identities of Dolgy-Kim-Kim for the Carlitz’s $q$-Bernoulli polynomials.

\section{Some identities of modified Carlitz’s $q$-Bernoulli polynomials}

For $n, m, w \in \mathbb{N} \cup \{0\}$, let us define the function $\widehat{T}_{n,m}(w \mid q)$ as follows:

$$\widehat{T}_{n,m}(w \mid q) = \sum_{i=0}^{w} [i]_q^m q^{ni} \quad (2.1)$$

In particular, we note that $T_{n,m}(0 \mid q) = \delta_{0,m}$ and

$$T_{n,0}(w \mid q) = \begin{cases} w + 1, & \text{if } n = 0, \\ [w+1]_q^n, & \text{if } n > 0. \end{cases}$$
From (1.3) and (1.9), we have
\[(q^{w_1 w_2})^{w_3} \int_{\mathbb{Z}_p} q^{-w_1 w_2(x+w_3)} e^{[w_1 w_2(x+w_3)]q} d\mu_{q^{w_1 w_2}}(x) - \int_{\mathbb{Z}_p} q^{-w_1 w_2} e^{[w_1 w_2]q} d\mu_{w_1 w_2}(x) \times e^{[w_1 w_2]q} d\mu_{q^{w_1 w_2}}(x) = t[w_1 w_2]_q \sum_{i=0}^{w_3-1} q^{w_1 w_2 i} e^{[w_1 w_2]q} \]
\[= \sum_{m=0}^{\infty} T_{1,m}(w_3 - 1 | q^{w_1 w_2}) [w_1 w_2]_q^{m+1} \frac{t^{m+1}}{m!}. \]  

On the other hand
\[q^{w_1 w_2 w_3} \int_{\mathbb{Z}_p} q^{-w_1 w_2(x+w_3)} e^{[w_1 w_2(x+w_3)]q} d\mu_{q^{w_1 w_2}}(x) \]
\[- \int_{\mathbb{Z}_p} q^{-w_1 w_2 x} e^{[w_1 w_2 x]q} d\mu_{w_1 w_2}(x) \]
\[= \sum_{m=0}^{\infty} \frac{[w_1 w_2]_q^m}{m!} \int_{\mathbb{Z}_p} q^{-w_1 w_2 x} (q^{w_1 w_2 w_3}[x+w_3]_q^m - [x]_q^m) d\mu_{q^{w_1 w_2}}(x) \]
\[= \sum_{m=0}^{\infty} \frac{[w_1 w_2]_q^m}{m!} (B_{m,q^{w_1 w_2}}(w_3) - B_{m,q^{w_1 w_2}}) \frac{t^m}{m!} \]
\[= \sum_{m=0}^{\infty} \frac{[w_1 w_2]_q^m+1}{m+1} \left( \frac{B_{m+1,q^{w_1 w_2}}(w_3) - B_{m+1,q^{w_1 w_2}}}{m+1} \right) \frac{t^{m+1}}{m!}. \]  

Therefore, by (2.2) and (2.3), we obtain the following theorem.

**Theorem 2.1.** For \(w_1, w_2, w_3 \in \mathbb{N}\) and \(m \geq 0\), we have
\[\frac{1}{m+1} (B_{m+1,q^{w_1 w_2}}(w_3) - B_{m+1,q^{w_1 w_2}}) = T_{1,m}(w_3 - 1 | q^{w_1 w_2}).\]

Now, we consider the following triple integrals:
\[I = q^{w_1 w_2 w_3} \int_{\mathbb{Z}_p^3} q^{-w_2 w_3 x_1 - w_1 w_3 x_2 - w_1 w_2 x_3} \]
\[\times e^{[w_2 w_3 x_1 + w_1 w_3 x_2 + w_1 w_2 x_3]q} d\mu_{q^{w_2 w_3}}(x_1) d\mu_{q^{w_1 w_3}}(x_2) \]
\[- \int_{\mathbb{Z}_p^3} q^{-w_2 w_3 x_1 - w_1 w_3 x_2 - w_1 w_2 x_3} \]
\[\times e^{[w_2 w_3 x_1 + w_1 w_3 x_2 + w_1 w_2 x_3]q} d\mu_{q^{w_2 w_3}}(x_1) d\mu_{q^{w_1 w_3}}(x_2) d\mu_{q^{w_1 w_2}}(x_3), \]  

which is obviously invariant under any permutation of \(w_1, w_2, w_3\).

We set
\[a = a_1(x) = q^{w_2 w_3(x_1 + w_1 y_1)}, \quad b = b(x_2) = q^{w_1 w_3(x_2 + w_2 y_2)}. \]
Then, by (2.4) and (2.5), we get

\[ I = \sum_{k,l=0}^{\infty} [w_2 w_3]_q^k [w_1 w_3]_q^l \frac{t^{k+l}}{k! l!} \int_{\mathbb{Z}_p^2} q^{-w_2 w_3 x_1 - w_1 w_3 x_2} a^l \]

\[ \times [x_1 + w_1 y_1]^k q^{w_2 w_3} [x_2 + w_2 y_2]^l q^{w_1 w_3} \left\{ \sum_{m=0}^{\infty} \frac{[w_1 w_2]_q^m (abt)_q^m}{m!} \right\} \]

\[ \times \int_{\mathbb{Z}_p} q^{-w_1 w_2 x_3} (q^{w_1 w_2 w_3} [x_3 + w_3]_q^{w_1 w_2} - [x_3]_q^m) d\mu_{q^{w_1 w_2}}(x_3) \]

\[ \times d\mu_{q^{w_2 w_3}}(x_1) d\mu_{q^{w_1 w_3}}(x_2). \quad (2.6) \]

From (2.2) and (2.3), the inner sum is

\[ \sum_{m=0}^{\infty} \frac{[w_1 w_2]_q^{m+1} (abt)_q^{m+1}}{m!} T_{1,m}(w_3 - 1 | q^{w_1 w_2}). \quad (2.7) \]

Thus, by (2.6) and (2.7), we get

\[ I = \sum_{k,l,m=0}^{\infty} [w_2 w_3]_q^k [w_1 w_3]_q^l [w_1 w_2]_q^{m+1} \frac{t^{k+l+m+1}}{k! l! m!} \]

\[ \times \int_{\mathbb{Z}_p} a^{l+m+1} [x_1 + w_1 y_1]_q^{w_2 w_3} q^{-w_2 w_3 x_1} d\mu_{q^{w_2 w_3}}(x_1) \]

\[ \times \int_{\mathbb{Z}_p} b^{m+1} [x_2 + w_2 y_2]_q^{w_1 w_3} q^{-w_1 w_3 x_2} d\mu_{q^{w_1 w_3}}(x_2). \quad (2.8) \]

Recovering \( a = q^{w_2 w_3 (x_1 + w_1 y_1)} \) and \( b = q^{w_1 w_3 (x_2 + w_2 y_2)} \), (2.8) can be rewritten as
\[ I = \sum_{n=0}^{\infty} \left\{ \sum_{k+l+m=n} \left( \begin{array}{c} n \\ k, l, m \end{array} \right) [w_2 w_3]_q^k [w_1 w_3]_q^l [w_1 w_2]_q^{m+1} T_{1,m}(w_3 - 1 \mid q^{w_1 w_2}) \right. \]
\[ \times q^{w_1 w_2 w_3(l+m+1) y_1} q^{w_1 w_2 w_3(m+1) y_2} \int_{\mathbb{Z}_p} q^{w_2 w_3(l+m) x_1} [x_1 + w_1 y_1]_q^k d\mu_{q^{w_2 w_3}}(x_1) \]
\[ \times \int_{\mathbb{Z}_p} q^{w_1 w_3 x_2} [x_2 + w_2 y_2]_q^l d\mu_{q^{w_1 w_3}}(x_2) \right\} \frac{t^{n+1}}{n!} \]
\[ = \sum_{n=0}^{\infty} \left\{ \sum_{k+l+m=n} \left( \begin{array}{c} n \\ k, l, m \end{array} \right) \beta_{k,q^{w_2 w_3}}^{(1,l+m+1)}(w_1 y_1) \beta_{i,q^{w_1 w_3}}^{(1,m+1)}(w_2 y_2) \right. \]
\[ \times T_{1,m}(w_3 - 1 \mid q^{w_1 w_2}) q^{w_1 w_2 w_3(l+m+1) y_1} q^{w_1 w_2 w_3(m+1) y_2} \]
\[ \times [w_2 w_3]_q^k [w_1 w_3]_q^l [w_1 w_2]_q^{m+1} \right\} \frac{t^{n+1}}{n!}, \] (2.9)

where the extended Carlitz’s q-Bernoulli polynomials \( \beta_n^{(k,h)}(x) \) are given by Kim to be

\[ \beta_n^{(k,h)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^{k} (h-i) y_i} [x + y_1 + \cdots + y_k]_q^n d\mu_q(y_1) \cdots d\mu_q(y_k) \]
\[ = \frac{1}{(1 - q)^n} \sum_{j=0}^{n} \left( \begin{array}{c} n \\ j \end{array} \right) (-1)^j q^{jx} \frac{(j + h)(j + h - 1) \cdots (j + h - k + 1)}{[j + h]_q [j + h - 1]_q \cdots [j + h - k + 1]_q}, \]

(see [3], [5], [9]).

At this expression is invariant under any permutation in \( w_1, w_2, w_3 \), we get the following theorem.

**Theorem 2.2.** For \( w_1, w_2, w_3 \in \mathbb{N} \) and \( n \geq 1 \), the following expressions

\[ \sum_{k+l+m=n-1} \left( \begin{array}{c} n \\ k, l, m \end{array} \right) \beta_{k,q^{w_2 w_3}}^{(1,l+m+1)}(w_2 w_3) \beta_{l,q^{w_1 w_3}}^{(1,m+1)}(w_2 w_3) \]
\[ \times T_{1,m}(w_3 - 1 \mid q^{w_1 w_2}) q^{w_1 w_2 w_3(l+m+1) y_1} q^{w_1 w_2 w_3(m+1) y_2} \]
\[ \times [w_2 w_3]_q^k [w_1 w_3]_q^l [w_1 w_2]_q^{m+1} \]

are the same for any \( \sigma \in S_3 \).

\( I \) can be rewritten as
where \( a = q^{w_3(x_1 + w_1 y_1)}, b = q^{w_3(x_2 + w_2 y_2)}. \)

From Theorem 2.1 and (2.10), we note that the inner integral is equal to

\[
abt[w_1 w_2]q \sum_{i=0}^{w_3-1} q^{w_1 w_2 i} e^{[w_1 w_2]q abt}.
\]

By (2.10) and (2.11), we get

\[
I = t[w_1 w_2]q \sum_{i=0}^{w_3-1} q^{w_1 w_2 i + w_1 w_2 w_3 (y_1 + y_2)} \int_{\mathbb{Z}_p^2} e^{[w_2 w_3]q [x_1 + w_1 y_1], w_2 w_3 t} \\
\times e^{[w_1 w_2]q x_2 + w_2 y_2 + \frac{w_2}{w_3} x_3} q^{w_1 w_3} d\mu_{q^{w_2 w_3}} (x_1) d\mu_{q^{w_1 w_3}} (x_2) \\
= t[w_2 w_1]q \sum_{i=0}^{w_3-1} q^{w_1 w_2 i + w_1 w_2 w_3 (y_1 + y_2)} \sum_{k,l=0}^{\infty} \frac{i^k l^l}{k! l!} q^{w_1 w_2 y_1} \sum_{k=0}^{n} q^{w_2 w_3 y_2} \sum_{k=0}^{n} \sum_{k=0}^{w_3-1} q^{w_1 w_2 i} \beta_{k,q^{w_2 w_3}} (w_1 y_1) q^{w_1 w_2 w_3 (n-k+1) y_1 + w_1 w_2 w_3 y_2} [w_1 w_2]q^{w_2 w_3} \\
\times [w_1 w_3]q^{n-k} q^{w_1 w_2 i} \beta_{n-k,q^{w_2 w_3}} (w_2 y_2 + \frac{w_2}{w_3} i) \frac{t^{n+1}}{n!}.
\]

As this expression is invariant under any permutation in \( w_1, w_2, w_3, \) we have the following results.
Theorem 2.3. For $w_1, w_2, w_3, n \in \mathbb{N}$, the following expressions
\[
\sum_{k=0}^{n-1} \binom{n-1}{k} \beta_{k,q}^{(1,l+1)} w_{\sigma(1)} y_1 q^{w_1 w_2 w_3 (y_2 + (n-k+1)y_1)}
\times [w_{\sigma(1)} w_{\sigma(2)}]_q [w_{\sigma(2)} w_{\sigma(3)}]_q [w_{\sigma(1)} w_{\sigma(3)}]_q \sum_{i=0}^{w_{\sigma(3)}-1} q^{w_{\sigma(1)} w_{\sigma(2)} i}
\times \beta_{n-k,q}^{w_{\sigma(1)} w_{\sigma(3)}} (w_{\sigma(2)} y_2 + \frac{w_{\sigma(2)}}{w_{\sigma(3)}} i)
\]
are all the same for any $\sigma \in S_3$.

References


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