On Modified $q$-Euler Polynomials under $S_5$

Jong Jin Seo

Department of Applied Mathematics
Pukyong National University
Busan 608-737, Republic of Korea

Taekyun Kim

Department of Mathematics
Kwangwoon National University
Seoul 139-701, Republic of Korea

Abstract

In [8], Seo-Kim derived some symmetric identities of the modified $q$-Euler polynomials under the symmetric group of degree four. In this paper, we consider some new identities of symmetry for the modified $q$-Euler polynomials under $S_5$ arising from fermionic $p$-adic integral on $\mathbb{Z}_p$.

Keywords: $q$-Euler polynomials under $S_5$

1. Introduction

Let $p$ be an odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of $p$-adic integers, the field of $p$-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The $p$-adic norm is normally defined by $|p|_p = \frac{1}{p}$ and let $q$ be an indeterminate such that $|1 - q|_p < p^{-1/p-1}$. The $q$-extension of number $x$ is defined as $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q \to 1}[x]_q = x$. As is well known, the Euler polynomials are defined by the generating function to be
\[
\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad \text{(see [1] - [8]).} \tag{1.1}
\]

When \( x = 0 \), \( E_n = E_n(0) \) are called the Euler numbers. From (1.1), we note that

\[
E_n(1) + E_n = 2\delta_{0,n}, \quad (n \geq 0),
\]

and

\[
E_n(x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} E_l, \quad \text{(see [2]).}
\]

Let \( C(\mathbb{Z}_p) \) be the space of continuous functions on \( \mathbb{Z}_p \). For \( f \in C(\mathbb{Z}_p) \), the fermionic \( p \)-adic integral on \( \mathbb{Z}_p \) is defined by Kim to be

\[
I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \quad \text{(see [7]).} \tag{1.2}
\]

From (1.2), we have

\[
I_{-1}(f_n) + (-1)^{n-1} I_{-1}(f) = 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l), \tag{1.3}
\]

where \( n \in \mathbb{N} \) and \( f_n(x) = f(x + n) \), (see [6], [7]).

Thus, by (1.3), we get

\[
\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad \text{(1.4)}
\]

The Carlitz’s \( q \)-Euler numbers are defined by

\[
\xi_{0,q} = 1, \quad q(q\xi_q + 1)^n + \xi_{n,q} = \left\{ \begin{array}{ll}
[2]_q, & \text{if } n = 0, \\
0, & \text{if } n > 0,
\end{array} \right. \tag{1.5}
\]

with the usual convention about replacing \( \xi^n_q \) by \( \xi_{n,q} \).

The Carlitz’s \( q \)-Euler polynomials are defined as

\[
\xi_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} q^{lx} \xi_{l,q}[x]_{q}^{n-l} = (q^x\xi_q + [x]_q)^n, \quad \text{(see [1], [5]).} \tag{1.6}
\]

In the viewpoint of (1.4), we consider the modified Carlitz’s \( q \)-Euler polynomials as follows :

\[
E_{n,q}(x) = \int_{\mathbb{Z}_p} [x + y]_{q}^{n} d\mu_{-1}(g), \quad (n \geq 0), \quad \text{(see [6], [8]).} \tag{1.7}
\]

When \( x = 0 \), \( E_{n,q} = E_{n,q}(0) \) are called the modified \( q \)-Bernoulli numbers.
From (1.3) and (1.7), we have
\[(qE_q + 1)^n + E_{n,q} = 2\delta_{0,n},\]
and
\[E_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{lx} E_{l,q}, \quad (n \geq 0),\]
with the usual convention about replacing \(E_q^n\) by \(E_{n,q}\).

In [8], Seo-Kim gave some interesting symmetric identities for the modified \(q\)-Euler polynomials under the symmetric group of degree four. The purpose of this paper is to give some new identities of symmetry for the modified \(q\)-Euler polynomials under the symmetric group of degree five arising from fermionic \(p\)-adic integral on \(\mathbb{Z}_p\).

2. Identities of \(E_{n,q}(x)\) under \(S_5\)

Let us assume that \(w_1, w_2, w_3, w_4, w_5 \in \mathbb{N}\) such that \(w_1 \equiv 1 \equiv w_2 \equiv w_3 \equiv w_4 \equiv w_5 (mod\ 2)\). Then we have

\[
\int_{\mathbb{Z}_p} e^{[w_1 w_2 w_3 w_4 y + w_1 w_2 w_3 w_4 w_5 x + w_5 w_4 w_3 w_2 i + w_5 w_4 w_3 w_2 j]} + \sum_{t=0}^{w_5-1} \sum_{t=0}^{w_5-1} (-1)^{m+y} e^{[w_1 w_2 w_3 w_4 (m+w_5 y) + w_1 w_2 w_3 w_4 x + w_5 w_4 w_3 w_2 i + w_5 w_4 w_3 w_2 j]} t \, d\mu_{-1}(y)
\]

Thus, by (2.1), we get

\[
\sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} \sum_{k=0}^{w_3-1} \sum_{l=0}^{w_4-1} (-1)^{i+j+k+l} \int_{\mathbb{Z}_p} e^{[w_1 w_2 w_3 w_4 y + w_1 w_2 w_3 w_4 w_5 x + w_5 w_4 w_3 w_2 i + w_5 w_4 w_3 w_2 j + w_5 w_4 w_3 w_2 k + w_5 w_4 w_3 w_2 l]} t \, d\mu_{-1}(y)
\]

As this expression is an invariant under any permutation \(\sigma \in S_5\), we have the following formula:
are the same for any $\sigma \in S_5$.

It is not difficult to show that

$$
\int_{\mathbb{Z}_p} e^{\left[w_1 w_2 w_3 w_4 y + \prod_{i=1}^{4} w_i\right] x} + w_5 w_4 w_3 w_2 i + w_5 w_4 w_3 w_1 j + w_5 w_4 w_1 w_2 k
$$

$$
= \sum_{n=0}^{\infty} \left[w_1 w_2 w_3 w_4\right]_q^n
$$

$$
\times \int_{\mathbb{Z}_p} \left[y + w_5 x + \frac{w_5}{w_1} i + \frac{w_5}{w_2} j + \frac{w_5}{w_3} k + \frac{w_5}{w_4} l\right]^n q^{w_1 w_2 w_3 w_4} d\mu_{1}(y)^{\frac{t^n}{n!}}
$$

From (2.4), we have

$$
\int_{\mathbb{Z}_p} \left[\prod_{m=1}^{4} w_m\right] y + \left[\prod_{m=1}^{5} w_m\right] x + w_5 w_4 w_3 w_2 i + w_5 w_4 w_3 w_1 j + w_5 w_4 w_1 w_2 k
$$

$$
= \left[w_1 w_2 w_3 w_4\right]_q^n E_{n,q} w_1 w_2 w_3 w_4 \left(w_5 x + \frac{w_5}{w_1} i + \frac{w_5}{w_2} j + \frac{w_5}{w_3} k + \frac{w_5}{w_4} l\right),
$$

where $n \geq 0$.

From (2.3) and (2.5), we have the following formula:

$$
\left[w_1 w_2 w_3 w_4\right]_q^n \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} \sum_{k=0}^{w_3-1} \sum_{l=0}^{w_4-1} (-1)^{i+j+k+l}
$$

$$
\times E_{n,q} w_1 w_2 w_3 w_4 \left(w_5 x + \frac{w_5}{w_1} i + \frac{w_5}{w_2} j + \frac{w_5}{w_3} k + \frac{w_5}{w_4} l\right)
$$

(2.6)
are the same for any $\sigma \in S_5$.

We observe that

$$
\left[ y + w_5 x + \frac{w_5}{w_1} i + \frac{w_5}{w_2} j + \frac{w_5}{w_3} k + \frac{w_5}{w_4} l \right]_{q^{w_1 w_2 w_3 w_4}}
$$

$$
= \frac{[w_5]_q}{[w_1 w_2 w_3 w_4]_q} [w_2 w_3 w_4 i + w_1 w_3 w_4 j + w_1 w_2 w_4 k + w_1 w_2 w_3 l]_{q^{w_5}}
$$

(2.7)

$$
+ q^{w_5 w_4 w_3 w_2 + w_5 w_4 w_3 w_1 + w_5 w_4 w_1 w_2 k + w_5 w_3 w_1 w_2 l} [y + w_5 x]_{q^{w_1 w_2 w_3 w_4}}.
$$

Thus, by (2.7), we get

$$
\left[ y + w_5 x + \frac{w_5}{w_1} i + \frac{w_5}{w_2} j + \frac{w_5}{w_3} k + \frac{w_5}{w_4} l \right]_{q^{w_1 w_2 w_3 w_4}}^n
$$

$$
= \sum_{m=0}^{n} \binom{n}{m} \left( \frac{[w_5]_q}{[w_1 w_2 w_3 w_4]_q} \right)^{n-m}
$$

$$
\times [w_2 w_3 w_4 i + w_1 w_3 w_4 j + w_1 w_2 w_4 k + w_1 w_2 w_3 l]_{q^{w_5}}^{n-m}
$$

$$
\times q^{m(w_5 w_4 w_1 w_3 + w_5 w_4 w_3 w_1 + w_5 w_4 w_1 w_2 k + w_5 w_3 w_1 w_2 l)} [y + w_5 x]_{q^{w_1 w_2 w_3 w_4}}^{m}.
$$

(2.8)

From (2.8), we can derive the following equation:

$$
\int_{\mathbb{K}} \left[ y + w_5 x + \frac{w_5}{w_1} i + \frac{w_5}{w_2} j + \frac{w_5}{w_3} k + \frac{w_5}{w_4} l \right]_{q^{w_1 w_2 w_3 w_4}}^n d\mu_{-1}(y)
$$

$$
= \sum_{m=0}^{n} \binom{n}{m} \left( \frac{[w_5]_q}{[w_1 w_2 w_3 w_4]_q} \right)^{n-m}
$$

$$
\times [w_2 w_3 w_4 i + w_1 w_3 w_4 j + w_1 w_2 w_4 k + w_1 w_2 w_3 l]_{q^{w_5}}^{n-m}
$$

$$
\times q^{m(w_5 w_4 w_1 w_3 + w_5 w_4 w_3 w_1 + w_5 w_4 w_1 w_2 k + w_5 w_3 w_1 w_2 l)} E_{m,q^{w_1 w_2 w_3 w_4}} (w_5 x).
$$

(2.9)

Hence, by (2.9), we get
\[ \left[ w_1 w_2 w_3 w_4 \right]_q^n \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} \sum_{k=0}^{w_3-1} \sum_{l=0}^{w_4-1} (-1)^{i+j+k+l} \times \int_{\mathbb{Z}} \left[ y + w_5 i + \frac{w_5}{w_1} j + \frac{w_5}{w_3} k + \frac{w_5}{w_4} l \right]^n \left\{ q^{w_1 w_2 w_3 w_4} I_{w_1 w_2 w_3 w_4} \right\} \, d\mu_{-1}(y) \]

\[ = \sum_{m=0}^{n} \binom{n}{m} \left[ w_1 w_2 w_3 w_4 \right]_q^{n-m} E_{m,q}^{w_1 w_2 w_3 w_4} (w_5 x) \times \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} \sum_{k=0}^{w_3-1} \sum_{l=0}^{w_4-1} (-1)^{i+j+k+l} \times q^{m(w_5 w_4 w_3 w_2 i + w_5 w_4 w_3 w_1 j + w_5 w_4 w_1 w_2 k + w_5 w_3 w_1 w_4 l)} \times \left[ w_2 w_3 w_4 i + w_1 w_3 w_4 j + w_1 w_2 w_4 k + w_1 w_2 w_3 l \right]^{n-m} \]

(2.10)

\[ \hat{T}_{n,q}(w_1, w_2, w_3, w_4 \mid m) \]

\[ = \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} \sum_{k=0}^{w_3-1} \sum_{l=0}^{w_4-1} (-1)^{i+j+k+l} q^{w_1 w_3 w_2 i + w_4 w_3 w_1 j + w_4 w_1 w_2 k + w_3 w_1 w_4 l} \times \left[ w_2 w_3 w_4 i + w_1 w_3 w_4 j + w_1 w_2 w_4 k + w_1 w_2 w_3 l \right]^{n-m}. \]

As this expression is an invariant under any permutation \( \sigma \in S_5 \), we have the following formula:

\[ \sum_{m=0}^{n} \binom{n}{m} \left[ w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)} w_{\sigma(4)} \right]_q^{m} \left[ w_{\sigma(5)} \right]_q^{n-m} E_{m,q}^{w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)} w_{\sigma(4)}} (w_{\sigma(5)} x) \times \hat{T}_{n,q}^{w_{\sigma(5)}} (w_{\sigma(1)}, w_{\sigma(2)}, w_{\sigma(3)}, w_{\sigma(4)} \mid m) \]

are the same for any \( \sigma \in S_5 \).

References

On modified $q$-Euler polynomials under $S_5$


Received: April 3, 2015; Published: June 21, 2015