Tree Cover of the Join and the Corona of Graphs

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Abstract

Let G be a graph and $\mathcal{T}_G = \{G_1, G_2, G_3, \dots, G_k\}$ be a collection of subgraphs of G where G_i is a subtree of G for every $i \in \{1, 2, \dots, k\}$. If for every edge $e \in E(G)$, there exists $G_i \in \mathcal{T}_G$ such that $e \in E(G_i)$, then $\mathcal{T}_G$ is a tree cover of G . The tree covering number of G is the minimum cardinality among the tree covers of G . In this paper, we establish some bounds for the tree covering numbers of the join and the corona of two vertex-disjoint graphs.

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1 Introduction

Let G be a simple graph. A subtree of G is a connected acyclic subgraph of G . A collection $\mathcal{T}_G = \{G_1, G_2, G_3, \dots, G_k\}$ of subgraphs of G is a tree cover of G if G_i is a subtree of G for every $i \in \{1, 2, \dots, k\}$ and for every edge $e \in E(G)$, there exists $G_i \in \mathcal{T}_G$ such that $e \in E(G_i)$. The *tree covering number* of G , denoted by $t_c(G)$, is given by $t_c(G) = \min\{|\mathcal{T}_G| : \mathcal{T}_G \text{ is a tree cover of } G\}$.

The graph G in Figure 1.1 has tree covering number equal to 2.

Figure 1.1: A graph G with $t_c(G) = 2$.

Indeed, let $\mathcal{T}_G = \{G_1, G_2\}$, where G_1 and G_2 are subgraphs of G shown in Figure 1.2 and Figure 1.3, respectively.

Figure 1.2: The graph G_1 .

Figure 1.3: The graph G_2 .

Clearly, G_1 and G_2 are subtrees of G . Moreover, $G_1 \cup G_2 = G$. Hence, every edge of G is either in G_1 or in G_2 . Consequently, $\mathcal{T}_G$ is a tree cover of G . Thus, by definition, $t_c(G) \leq |\mathcal{T}_G| = 2$. Since G is not a tree, Theorem 2.2 found in [1] asserts that $t_c(G) \geq 2$. Combining the two inequalities gives $t_c(G) = 2$.

The next section establishes an upper bound for the tree covering number of the join of two vertex-disjoint graphs.

2 Tree Covering Number of the Join of Graphs

Here, we formally define the join of two vertex-disjoint graphs.

Definition 2.1 [4] Let G and H be vertex-disjoint graphs. The *join* $G \oplus H$ of G and H has vertex-set $V(G \oplus H) = V(G) \cup V(H)$ and edge-set

$$E(G \oplus H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$$

Consequently,

$$|V(G \oplus H)| = |V(G)| + |V(H)|,$$

and

$$|E(G \oplus H)| = |E(G)| + |E(H)| + |V(G)||V(H)|.$$

Note also that the operation $\oplus$ is commutative, i.e., $G \oplus H$ is isomorphic to $H \oplus G$ with respect to adjacency.

Let us consider an illustration of the above definition.

Example 2.2 Consider the path P_9 and the complete graph K_1 with vertex-set $V(K_1) = \{u\}$. Then the join of P_9 and K_1 is shown below.

Figure 2.1: The join $P_9 + K_1$.

In the above illustration, every vertex of P_9 is joined with the vertex u .

An upper bound for the tree covering number of the join of two vertex-disjoint graphs is established in the following theorem.

Theorem 2.3 *Let G and H be two vertex-disjoint graphs of orders m and n , respectively. Then,*

$$t_c(G \oplus H) \leq \min\{m + t_c(H), n + t_c(G)\}.$$

Proof: Let $V(G) = \{v_1, v_2, v_3, \dots, v_m\}$ and $V(H) = \{u_1, u_2, u_3, \dots, u_n\}$. Then $v_i u_j \in E(G \oplus H)$ for all $i = 1, 2, 3, \dots, m$ and for all $j = 1, 2, 3, \dots, n$. Let $\mathcal{T}_G$ and $\mathcal{T}_H$ be tree covers of G and H , respectively, such that $|\mathcal{T}_G| = t_c(G)$ and $|\mathcal{T}_H| = t_c(H)$. Now, for every vertex $v \in V(G)$, $\langle\{v\}\rangle \oplus \overline{H \cup \overline{H}}$ is a star, and hence a tree. Moreover, there exists $G_v \in \mathcal{T}_G$ with $v \in V(G)$ and $G_v \cup \langle\{v\}\rangle \oplus \overline{H \cup \overline{H}}$ is a tree. Let

$$\mathcal{T}_{G+H} = \mathcal{T}_H \cup \{G_v \cup \langle\{v\}\rangle \oplus \overline{H \cup \overline{H}} : v \in V(G)\}.$$

Then $\mathcal{T}_{G+H}$ is a tree cover of $G \oplus H$. Thus,

$$t_c(G \oplus H) \leq |\mathcal{T}_H| + |\{G_v \cup \langle\{v\}\rangle \oplus \overline{H \cup \overline{H}} : v \in V(G)\}|.$$

But $|\mathcal{T}_H| = t_c(H)$ and $|\{G_v \cup \langle\{v\}\rangle \oplus \overline{H \cup \overline{H}} : v \in V(G)\}| = |V(G)| = m$. Thus,

$$t_c(G \oplus H) \leq m + t_c(H). \quad (1)$$

Similarly,

$$t_c(G \oplus H) \leq n + t_c(G). \quad (2)$$

Combining Inequalities (1) and (2) gives the desired result. $\square$

Consider now the complete bipartite graph $K_{m,n}$. An upper bound for its tree covering number is established in the following theorem.

Theorem 2.4 *Let m and n be positive integers. Then, $t_c(K_{m,n}) \leq \min\{m, n\}$.*

Proof: Note that $K_{m,n} = \overline{K_m} \oplus \overline{K_n}$. Thus by Theorem 2.3,

$$\begin{aligned} t_c(K_{m,n}) &= t_c(\overline{K_m} \oplus \overline{K_n}) \\ &\leq \min\{m + t_c(\overline{K_n}), n + t_c(\overline{K_m})\} \\ &= \min\{m + 0, n + 0\} \\ &= \min\{m, n\}. \end{aligned}$$

Thus, $t_c(K_{m,n}) \leq \min\{m, n\}$. $\square$

The tree covering number of the corona of graphs is established in the following section.

3 Tree Covering Number of the Corona of Graphs

Formally, we define the corona of two vertex-disjoint graphs.

Definition 3.1 [4] The *corona* $G \circ H$ of two graphs G and H is the graph obtained by taking one copy of G of order n and n copies of H , and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H .

Example 3.2 The figure below illustrates the corona $P_3 \circ C_3$.

Figure 3.1: The corona $P_3 \circ C_3$.

Note that if H is a tree, then $G \circ H$ has tree covering number equal to the tree covering number of G . We formally write this result in the following theorem.

Theorem 3.3 Let G be a connected graph. Then for any tree T , $t_c(G \circ T) = t_c(G)$.

Next, we give an upper bound for the tree covering number of the corona of two graphs as a linear combination of the tree covering number of G and the tree covering number of H .

Theorem 3.4 Let G and H be two nontrivial connected graphs of orders m and n , respectively. Then $t_c(G \circ H) \leq t_c(G) + mt_c(H)$.

Proof: Let $\mathcal{T}_G$ and $\mathcal{T}_H$ be tree covers of G and H , respectively, such that $|\mathcal{T}_G| = t_c(G)$ and $|\mathcal{T}_H| = t_c(H)$. For every $u \in V(G)$, each copy H_u of H can be covered by $|\mathcal{T}_H|$ subtrees. Hence, all the copies of H in $G \circ H$ can be covered by $mt_c(H)$ subtrees. Now, for every vertex $v \in V(G)$, there exists $G_v \in \mathcal{T}_G$ with $v \in G_v$ and $G_v \cup \langle \{v\} \rangle \oplus \overline{H \cup \overline{H}}$ is a tree. Thus, the family $\mathcal{T}_{G \circ H} = \bigcup \{ \mathcal{T}_{H_u} : u \in V(G) \} \cup \{ G_v \cup \langle \{v\} \rangle \oplus \overline{H \cup \overline{H}} \}$ is a tree cover of $G \circ H$. Accordingly,

$$\begin{aligned} t_c(G \circ H) &\leq \left| \bigcup \{ \mathcal{T}_{H_u} : u \in V(G) \} \cup \{ G_v \cup \langle \{v\} \rangle \oplus \overline{H \cup \overline{H}} \} \right| \\ &= m|\mathcal{T}_H| + |\mathcal{T}_G| \\ &= mt_c(H) + t_c(G). \end{aligned}$$

This completes the proof. $\square$

References

- [1] R.G. Artes, Jr. and R.D. Dignos, Tree Cover of Graphs, *Applied Mathematical Sciences*, **8** (201), 7469-7473.
<http://dx.doi.org/10.12988/ams.2015.49728>
- [2] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, New York: Chapman and Hall, 1996.
- [3] G. Chartrand and O.R. Oellermann, *Applied and Algorithmic Graph Theory*, New York: McGraw-Hill, Inc., 1993.
- [4] F. Harary, *Graph Theory*, Addison Wesley Publishing Co. Inc., 1972.
- [5] K. Thulasiraman and M.N.S Swamy, *Graph Theory and Algorithms*, A Wiley-Interscience Publication, Canada, 1992.
<http://dx.doi.org/10.1002/9781118033104>

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