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A Polynomial Expansion of Axial Velocity Profiles to Solve Transient Laminar Flow in Elastic Pipe

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Abstract

This study provides a theoretical and numerical modelling of shear stress due to the friction of transient laminar flow on pipe wall. This work is a simplified model of Prado et al. [1]. It is based on the expansion of the instantaneous velocity profiles of the flow in polynomial series in time and radial variable across the section of pipe. The set partial derivatives equations obtained from the conservation of mass and the theorem of momentum is then solved by the method of characteristics. The results obtained are in good agreement with those of Holmboe and Rouleau [2], in steady state Newtonian laminar flow.

Keywords: Transient Shear stress, Newtonian laminar flow, Velocity profiles, Polynomial expansion, Method of characteristics

Notations

A : section of pipe
a : celerity of water hammer
a_j : coefficient of polynomial expansion
$D(x,0)$: initial inner diameter of the pipe
$D(x,t)$: inner diameter of the pipe at time t
e : thickness of the pipe wall
$D_m = D + e$: average diameter
E : Young's modulus of the pipe wall
g : acceleration of gravity
G_{ij} : matrix
H_{ij} : matrix
H_0 : piezometric head at pipe at the reservoir
i: index
j: index
p, P : pressure, average pressure
q : heat flux
r : radial variable
$\bar{r}$: dimensionless variable
s :entropy
S_i : source term
v_x : instantaneous axial velocity
V : average velocity
V_i : weight average velocity
κ : bulk modulus of water
μ :dynamic viscosity of the fluid
ν : kinematic viscosity
α : anchored coefficient of the pipe
ρ : density of water
τ_p : global shear stress
t: time
T: temperature

1 Introduction

In transient pipe flow, the essential part of the energy dissipation comes from the pressure loss due to friction of the fluid on the wall of the pipe. The friction in transient pipe flow differs from the friction for steady state pipe flow, Streeter and Wylie [3, 4]. The discrepancies are introduced by a difference in velocity profile, turbulence and transition from laminar to turbulent flow and vice versa. There are number of unsteady friction models which have been proposed in the literature. We can, principally, cite the works of Zielke [5], Trikha [6], Kagawa et al. [7], Brown [8], Suzuki et al. [9], Vardy et al. [10].
In this works, the friction term is dependent on instantaneous mean flow velocity V and weights for past velocity changes.
In a different way, Brunone et al. [11], has expressed the friction term dependent function of instantaneous mean flow velocity V, instantaneous local acceleration and instantaneous convective acceleration. Generally, difficulties arise in analysis of transient turbulent flow. Models from the literature are calibrated for certain flow conditions whereas the development of a general friction model in transient turbulent flow is a subject of intensive research worldwide.
Compared to these one-dimensional models Vardy et al. [11] developed a two-dimen-

sional model. It has been shown that, contrary to the quasi-stationary regime, the velocity profiles are not parabolic. Recently, to evaluate the wall shear stress in a transient laminar flow in pipes, Prado et al. [1] have developed a semi-analytical method based, essentially, on the polynomial series expansion of instantaneous axial velocity profiles as function of radial variable and time in a section of pipe. This method is associated with the method of characteristics.
This work is, essentially, devoted to a relatively simple adaptation of this model to calculate the velocity profiles, pressure, mean flow velocity and wall shear stress in the transient pipe. The set differential equations obtained are of hyperbolic type and suitable to be resolved by the method of characteristics.

2 Assumptions and basic equations

This study is conducted under the assumption of axisymmetric unsteady flow of Newtonian, isentropic and compressible fluid. The deformation of the pipe wall is of low amplitude. The inertia terms are negligible and the pipe is modelled by a juxtaposition of independent rings without mass. The radius of the pipe is sufficiently negligible compared with its length enough that the current lines of fluid are straight Assume, furthermore, that the longitudinal velocity gradients are very low compared with the transverse gradients.
The basic relations for the fluid are:

- Conservation of mass:

$$\frac{\partial \rho}{d t} + \frac{\partial(\rho v_x)}{\partial x} = 0 \tag{1}$$

- momentum equation:

$$\rho \frac{d v_x}{d t} + \frac{\partial p}{\partial z} = \frac{\mu}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v_x}{\partial r}\right) \tag{2}$$

$$\frac{\partial p}{\partial r} = 0 \tag{3}$$

Relation (3) shows that the pressure is substantially constant in each section of the pipe and is equal to its average value. The instantaneous deformation of the pipe is related to the pressure by the Hooke’s law [4]:

$$\frac{D(x,t) - D(x,0)}{D(x,0)} = \frac{p\, D(x,0)}{2\, e\, E}$$

By introducing the average velocity of the flow across a section of pipe:

$$V = \int_A v_x dA / \int_A dA = \int_0^R v_x r dr / \int_0^R r dr = 2\int_0^1 v_x \bar{r}\, d\bar{r}$$

Where we noted by the reduced variable. $\bar{r} = r / R$

The integrating of equations (1) and (2) on a cross section A of the pipe provides the following system of partial differential equations:

$$\frac{\partial(\rho A)}{\partial t} + \frac{\partial(\rho AV)}{\partial x} = 0 \tag{4}$$

$$\rho \frac{\partial V}{\partial t} + \frac{\partial p}{\partial x} = - 4 \frac{\tau_p}{D} \tag{5}$$

The wall shear stress τ_p depends on the velocity profiles

- Energy equation

By using entropy as a state variable to express the energy of the fluid, we have the following relation differential:

$$\rho \frac{ds}{dt} = \frac{q}{ST} - 4 \frac{V \tau_p}{D T} \tag{6}$$

The thermodynamic behaviour of the fluid results in the relationship

$$\rho = \rho(s, p)$$

Which, by differentiation can be written in the form:

$$\frac{d\rho}{\rho} = \frac{dp}{\kappa} - \frac{\beta T}{c_p} ds \tag{7}$$

Where we put:

$$\frac{1}{\kappa} = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_s \tag{8}$$

and

$$\frac{\beta T}{c_P} = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial s}\right)_P \tag{9}$$

κ, c_P and β are respectively, the fluid compressibility, coefficient at constant entropy, the specific heat and coefficient of volumetric expansion at constant pressure.
In the case of an isentropic flow, we can obtain the following relation:
specific heat and coefficient of volumetric expansion at constant pressure.
In the case of an isentropic flow, we can obtain the following relation:

$$\frac{d\rho}{\rho} = \frac{dp}{\kappa} \tag{10}$$

This leads, finally, to the following differential system to solve:

$$\frac{\partial p}{\partial t} + \rho a^2 \frac{\partial V}{\partial x} = 0 \tag{4a}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\partial V}{\partial t} = S_0 \tag{4b}$$

Where the source term S_0 is related to the wall shear stress by:

$$S_0 = -4\,\tau_p / \rho\, D$$

$a = \left(\rho\left(1/\kappa + \alpha\, D_m / eE\right)\right)^{-1/2}$ waterhammer celerity

3 Modelling the wall stress

The system (4a) and (5a) of partial differential equations is hyperbolic and is perfectly suitable to be resolved by the method of characteristics. To solve it, it is first necessary to know the wall shear stress, which itself depends of the velocity profiles. The technique of calculation proposed by Prado et al.[1] is to model the velocity profile in each section of pipe and at each time by a polynomial term of the dimensionless variable.

$\bar{r} = r / R$ in the form:

$$v_x(x,\bar{r},t) = \sum_{j\in J} a_j(x,t)\,(1-\bar{r}^{\,j}) \tag{11}$$

Where the summation extends to an arbitrary set of integer numbers greater or equal to 2. The instantaneous velocity as defined, verify, automatically, the condition of adhesion at the wall represented by the reduced variable $\bar{r} = 1$ et should be able to represent, also, the condition of the stationary laminar flow of Poiseuille - Hagen:

$$V_{HP} = 2V\,(1-\bar{r}^{\,2}) \tag{12}$$

3.1 Determination of the coefficients

By introducing the weighted average velocities such as:

$$V_i(x,t) = \frac{\int_A v_x(x,r,t)\, r^i\, dr}{\int_A r^i\, dA} = \frac{\int_0^1 v_x(x,\bar{r},t)\,\bar{r}^{\,i+1}\, d\bar{r}}{\int_0^1 \bar{r}^{\,i+1}\, d\bar{r}} = (2+i)\int_0^1 v_x(x,\bar{r},t)\,\bar{r}^{\,i+1}\, d\bar{r}$$

For $i = 0, 1 .... \dim(J-1)$, the expressions are obtained for the coefficients a_j :

$$a_j = \sum_{i=0}^{\dim(J)-1} G_{ij}^{-1}\, V_i \tag{13}$$

Where, considering the expression (11) of the velocity profiles,

$$G_{ij} = (i+2)\int_0^1 (1-\bar{r}^{\,j})\,\bar{r}^{\,i+1}\,d\bar{r} = 1 - \frac{i+2}{i+j+2} \tag{14}$$

From the expression (11) velocity profile, it is also possible, to deduce for a laminar flow of Newtonian fluid, the wall stress in the form:

$$\tau_p = \mu \frac{\partial v_x(x,R,t)}{\partial r} = -\frac{\mu}{R}\sum_{i=0}^{\dim(J)-1}\left[\sum jG_{ij}^{-1}\right]V_i \tag{15}$$

Therefore, This model considers the wall stress, τ_p, not only, according to the average speed as in the case of quasi-stationary model, but also function weighted average velocities V_i, $i = 1,,2,.....\dim(J)-1$.

3.2 Calculation of weighted average velocities

Integrating equation (2) on a right section of the pipe after multiplication by r^i, for $i = 0,.....\dim(J)-1$, leads to differential system:

$$\frac{\partial V_i}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial x} = S_i \tag{16}$$

Where the source terms

$$S_i = -\frac{2\tau_i}{\rho R} = \nu \frac{\int_A \frac{\partial}{\partial r}\left(r\frac{\partial v_x}{\partial r}\right) r^{i-1}dr}{\int_A r^i dr} =$$

(17)

$$\frac{(i+2)\nu}{R^2}\int_0^1 \frac{\partial}{\partial \bar{r}}\left(\bar{r}\,\frac{\partial v_x}{\partial \bar{r}}\right)\bar{r}^{\,i}\,d\bar{r} = \sum_{j\in J}\sum_{k=0}^{\dim(J)-1} H_{ij}G_{jk}^{-1}V_k$$

The coefficients of the matrix H are defined by:

$$H_{ij} = -\frac{\nu}{R^2}\frac{(i+2)\,j^2}{(i+j)} \tag{18}$$

ν : Kinematic viscosity of the fluid, τ_i the weighted wall stress and wall shear stress at the wall is:

$$\tau_p = -\frac{\rho R}{2} S_0 \tag{19}$$

In summary, we have to solve the following set of derivative partial equations:

$$\frac{\partial p}{\partial t} + \rho a^2 \frac{\partial V}{\partial x} = 0 \tag{20}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\partial V}{\partial t} = S_0 \tag{21}$$

$$\frac{\partial V_i}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial x} = S_i \tag{22}$$

4 Modelling the wall stress

The numerical solution of hyperbolic systems (19), (20) and (21) associated to initial and boundary conditions can easily be obtained by the usual method of characteristics [16], [17]. The system can be easily integrated by the method of finite differences along the characteristic curves slope $\pm a$ et 0. This leads to the following algebraic relationship, to calculate the values of the various parameters at time n +1 as a function of those at time n:

$$V_l^{n+1} - V_{l-1}^n + \frac{1}{\rho a}(P_l^{n+1} - P_{l-1}^n) = S_0\Big|_{l-1}^n \Delta t \tag{23}$$

Along the characteristic of slope $dx/dt = +a$

$$V_l^{n+1} - V_{l+1}^n + \frac{1}{\rho a}(P_l^{n+1} - P_{l+1}^n) = S_0\Big|_{l+1}^n \Delta t \tag{24}$$

Along the characteristic $dx/dt = -a$
and,

$$V_i\Big|_l^{n+1} - V_i\Big|_l^n - \left(V_0\Big|_l^{n+1} - V_i\Big|_l^n\right) = \left(S_i\Big|_l^n - S_0\Big|_l^n\right)\Delta t \tag{25}$$

for *i = 1,2,....dim(J)-1*, along the characteristic $dx/dt = 0$
Where the indices n, *l* are respectively the discretization in time and space.

5 Initial and boundary conditions

The initial conditions is a fully established a steady and laminar flow of Reynolds number less than 2100, and the balance for the pipe wall. At time t = 0, the valve is suddenly closed. The boundary conditions are in addition to the pressure imposed by the tank on the upstream end, the instantaneous closing of valve on the downstream.

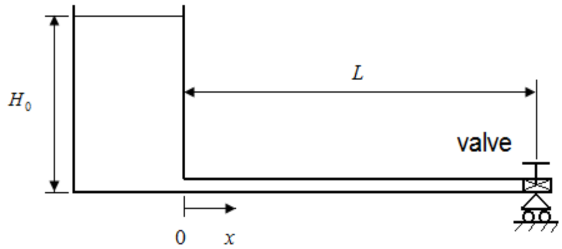

Fig. 1: diagram of the system studied

6 Application and results

In order to illustrate the results and validate this model, we consider the relevant parameters of the experience carried out by Holmboe and Rouleau [2], corresponding to the situation shown in figure 2, above, in which:
Pipe length, 36m
Pipe radius, $R = 0.0127m$
Kinematic viscosity, $\nu = 3.96 \times 10^{-5} m^2 / s$ $\quad (at\ 27°C)$
Water hammer celerity, $a = 1324.4 m/s$

The steady state velocity before closing the valve, $V_0 = 0.128 m/s$
Reynolds number, $\mathrm{Re} = 82.1$
The height of the fluid in the reservoir, supposed constant, $H_0(m)$
The problem was solved for three options of the set J.
a/ $J_a = \{2, 6, 10, 12\}$ dim(J_a)-1 = 3, i=0,1,2,3
b/ $J_b = \{2, 3, 4, 6, 8, 9, 10, 11\}$ dim(J_b)=8, i=0,1,2,3,4,5,6,7
c/ $J_c = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ *dim(J_b)=10, i=0,1,2,3,4,5,6,7,8,9,10*
The expressions for the shear stress τ_p at the wall for the three cases are, respectively:
option a:

$$S0 = \frac{\nu}{R^2}(-793{,}3306771V_0+4295{,}737357V_1-7113{,}951694V_2+3687{,}261097V_3)$$

$$S1 = \frac{\nu}{R^2}(-1206{,}237804V_0+6472{,}605724V_1-10691{,}18644V_2+5535{,}348455V_3)$$

$$S2 = \frac{\nu}{R^2}(-1594{,}857502V_0+8592{,}328612V_1-14229{,}07735V_2+7375{,}03951V_3)$$

$$S3 = \frac{\nu}{R^2}(-1984{,}673416V_0+10730{,}34176V_1-17793{,}31498V_2+9221{,}9527V_3)$$

$$\tau_p = \frac{\mu}{R}(-396{,}6653385V_0+2147{,}868678V1-3556{,}975847V_2+1843{,}630548V_3)$$

Due to the complexity of the expressions of the terms S_i for the other options, we have limited ourselves arbitrarily to the expression of wall shear stress and this leads to:

<u>option: b</u>

$$\tau_p = \frac{\mu}{R}(-266,428492V_0+1031,58272V_1-747,207996V_2-654,815V_3+23,770713V_4 -196,895888V_5 + 417,029353V_6 - 65,4128372V_7)$$

<u>Option c:</u>

$$\tau_p = \frac{\mu}{R}(-365,6916525V_0+1633,964536V_1-1789,279339V_2-445,0024444V3+ 718,1363099V4+620,7349056V5+898,6164678V6-1719,144071V_7-440,2781301 V_8 +933,1216069V_9)$$

In these applications, we analysis the response to fast closing of the valve at the downstream of the pipe. The figures 2, 3 and 4 correspond to the evolution versus time, of the non-dimensional presentations of the pressure and the mean velocity of the flow at the pipe midpoint and at the valve downstream of the pipe. The graphs in the figure 5 illustrate, in the same conditions, the evolution of the non-dimensional stress at the median of the pipe. The results for each figure show the superposition of different options against the quasi stationary state. One can notice the clear difference between the quasi-stationary model and the model presented. However, we can see from the graphs that the average velocity and pressure are not dependent upon the order of the polynomial approximation of profiles velocity. These results are, qualitatively, compared to those found by Holmboe et al. and are in perfect agreement.

The figure 6 shows that this model, allows, in addition to the pressure, average velocity and stress to have more information on the velocity profiles in transient laminar flow in pipe.

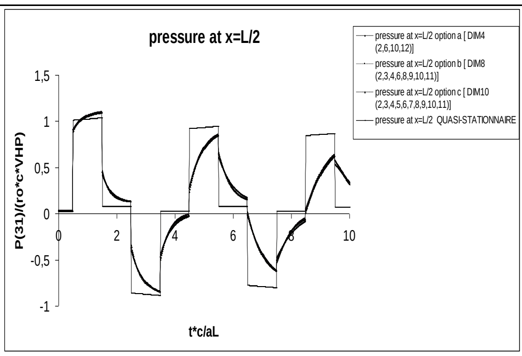

Figure.2: Pressure at pipe midpoint

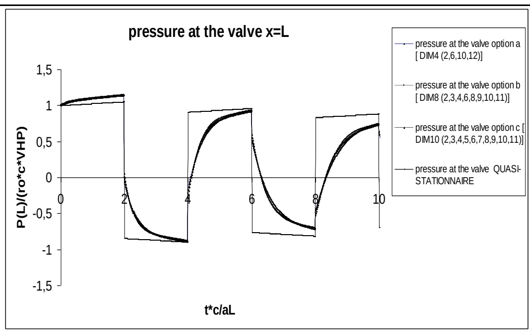

Figure.3: Pressure at the valve

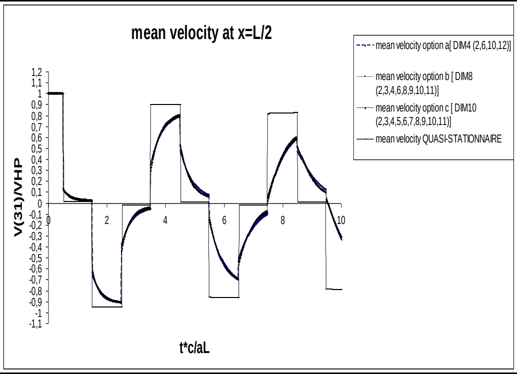

Figure.4: Mean velocity at pipe midpoint

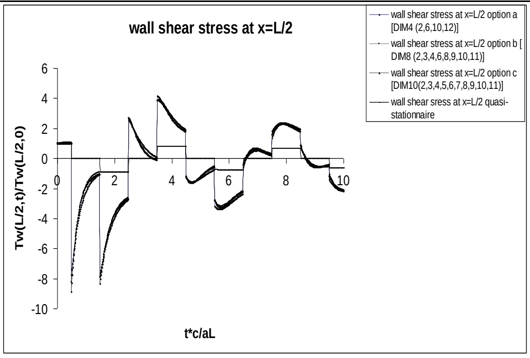

Figure.5: Stress at the pipe midpoint

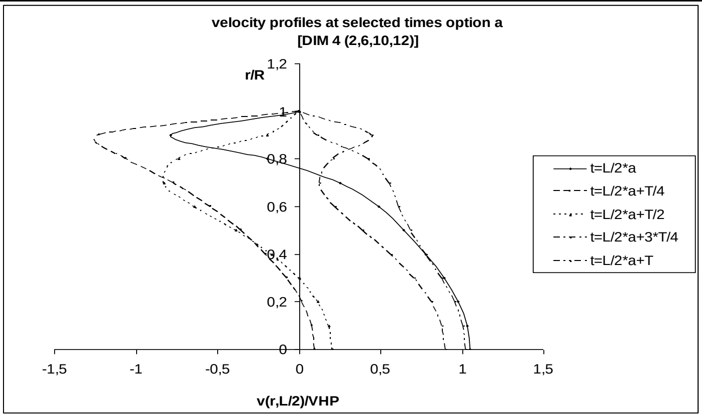

Figure.6: Profiles velocity at the pipe midpoint

7 Conclusion

This study point out that, compared to the quasi-stationary flow model, this model taking into account the variation of velocity profiles has advantage to introduce a significant correction to the transient stress of flow in pipe. However, we find that the increasing the degree polynomial expansion of velocity profiles has effects on the velocity profiles but not on the mean velocity, pressure and stress of flow on pipe. All results for each of the chosen sets are close to each other.

This model has the advantage to require little additional terms to give a better representation of the wall shear stress and, thus, it can easily be used alongside the existing codes to calculate the laminar transient flow in pipes.

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