Solutions of Generalized Linear Matrix Differential Equations which Satisfy Boundary Conditions at Two Points

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Abstract

In this article, we study a boundary value problem of a class of generalized linear matrix differential equations whose coefficients are square constant matrices. By using matrix pencil theory we obtain formulas for the solutions and we give necessary and sufficient conditions for existence and uniqueness of solutions. Moreover we provide some numerical examples. These kinds of systems are inherent in many physical and engineering phenomena.

Keywords: linear differential equations, boundary value problem, singular, system

1 Introduction

Many authors have studied generalized continuous & discrete time systems, see [3-6, 10-12, 14-16, 19, 23, 24, 29-34, 37-39, 44-46], and their applications, see [2, 3, 20-22, 25, 26, 48-50, 53, 54]. Many of these results have already been extended to systems of differential & difference equations with fractional operators, see [13, 17, 18, 20, 24, 25, 33-36, 40-42]. In this article, our purpose is to study the solutions of a generalized boundary value problem of Linear Matrix Differential Equations (LMDEs) into the mainstream of matrix pencil

theory. A boundary-value problem consists of finding solutions which satisfies an ordinary matrix differential equation and appropriate boundary conditions at two or more points. Thus we consider the boundary value problem

$$FY'(t) = GY(t), \quad AY(t_0) = D, \quad BY(t_N) = E. \quad (1)$$

where $F, G, A, B \in \mathcal{M}(m \times m; \mathcal{F})$ and $Y(t), D, E \in \mathcal{M}(m \times 1; \mathcal{F})$, (i.e. the algebra of square matrices with elements in the field $\mathcal{F}$). For the sake of simplicity we set $\mathcal{M}_m = \mathcal{M}(m \times m; \mathcal{F})$ and $\mathcal{M}_{nm} = \mathcal{M}(n \times m; \mathcal{F})$. Systems of type (1) are more general, including the special case when $F = I_n$, where I_n is the identity matrix of $\mathcal{M}_n$.

Mathematical Background and Notation

This brief subsection introduces some preliminary concepts and definitions from matrix pencil theory, which are being used throughout the paper. Linear systems of type (1) are closely related to matrix pencil theory, since the algebraic geometric, and dynamic properties stem from the structure by the associated pencil $sF - G$. Given $F, G \in \mathcal{M}_{nm}$ and an indeterminate $s \in \mathcal{F}$, the matrix pencil $sF - G$ is called regular when $m = n$ and $\det(sF - G) \neq 0$, see [10, 28, 38, 43, 47]. In any other case, the pencil will be called singular. In addition, the pencil $sF - G$ is said to be *strictly equivalent* to the pencil $s\tilde{F} - \tilde{G}$ if and only if there exist non-singular $P \in \mathcal{M}_m$ and $Q \in \mathcal{M}_n$ such as

$$P(sF - G)Q = s\tilde{F} - \tilde{G}.$$

In this article, we consider the case that pencil is *regular*. The class of $sF - G$ is characterized by a uniquely defined element, known as a complex Weierstrass canonical form, $sF_w - G_w$, see [10, 28, 38, 43, 47], specified by the complete set of invariants of the pencil $sF - G$. This is the set of *elementary divisors* (e.d.) obtained by factorizing the invariant polynomials $f_i(s, \hat{s})$ into powers of homogeneous polynomials irreducible over field F . In the case where $sF - G$ is a regular, we have e.d. of the following type:

- e.d. of the type s^p are called *zero finite elementary divisors* (z. f.e.d.)
- e.d. of the type $(s - a)^\pi$, $a \neq 0$ are called *nonzero finite elementary divisors* (nz. f.e.d.)
- e.d. of the type $\hat{s}^q$ are called *infinite elementary divisors* (i.e.d.).

Let $B_1, B_2, \dots, B_n$ be elements of $\mathcal{M}_n$. The direct sum of them denoted by $B_1 \oplus B_2 \oplus \dots \oplus B_n$ is the block $\text{diag}\{B_1, B_2, \dots, B_n\}$. Then, the complex

Weierstrass form $sF_w - G_w$ of the regular pencil $sF - G$ is defined by $sF_w - G_w := sI_p - J_p \oplus sH_q - I_q$, where the first normal Jordan type element is uniquely defined by the set of f.e.d.

$$(s - a_1)^{p_1}, \dots, (s - a_\nu)^{p_\nu}, \quad \sum_{j=1}^{\nu} p_j = p$$

of $sF - G$ and has the form

$$sI_p - J_p := sI_{p_1} - J_{p_1}(a_1) \oplus \dots \oplus sI_{p_\nu} - J_{p_\nu}(a_\nu).$$

And also the q blocks of the second uniquely defined block $sH_q - I_q$ correspond to the i.e.d.

$$\hat{s}^{q_1}, \dots, \hat{s}^{q_\sigma}, \quad \sum_{j=1}^{\sigma} q_j = q$$

of $sF - G$ and has the form

$$sH_q - I_q := sH_{q_1} - I_{q_1} \oplus \dots \oplus sH_{q_\sigma} - I_{q_\sigma}.$$

Thus, H_q is a nilpotent element of $\mathcal{M}_n$ with index $\tilde{q} = \max\{q_j : j = 1, 2, \dots, \sigma\}$, then

$$H_q^{\tilde{q}} = 0_{q,q}.$$

We denote with $O_{q,q}$ the zero matrix. $I_{p_j}, J_{p_j}(a_j), H_{q_j}$ are defined as

$$J_{p_j}(a_j) = \begin{bmatrix} a_j & 1 & \dots & 0 & 0 \\ 0 & a_j & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & a_j & 1 \\ 0 & 0 & \dots & 0 & a_j \end{bmatrix} \in \mathcal{M}_{p_j}, H_{q_j} = \begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix} \in \mathcal{M}_{q_j}.$$

2 Main Results-Solution space form of a consistent boundary value problem

In this section, the main results for a consistent boundary value problem of type (1) are analytically presented. Moreover, it should be stressed out that these results offer the necessary mathematical framework for interesting applications.

Definition 2.1. The boundary value problem (1) is said to be consistent if it possesses at least one solution.

Consider the problem (1). From the regularity of $sF - G$, there exist non-singular $\mathcal{M}(m \times m, F)$ matrices P and Q such that (see also Section 1),

$$PFQ = F_w = I_p \oplus H_q, \quad PGQ = G_w = J_p \oplus I_q \quad (2)$$

Where $I_{p_j}, J_{p_j}(a_j), H_{q_j}$ are defined in Section 1.

Lemma 2.1. System (1) is divided into two subsystems:

$$Z'_p(t) = J_p Z_p(t), \quad (3)$$

and the subsystem

$$H_q Z'_q(t) = Z_q(t). \quad (4)$$

Proof. Consider the transformation

$$Y(t) = QZ(t).$$

Substituting the previous expression into (1) we obtain

$$FQZ'(t) = GQZ(t),$$

where by multiplying by P and using (2) we arrive at

$$F_w Z'(t) = G_w Z(t).$$

Moreover, we can write $Z(t)$ as $Z(t) = \begin{bmatrix} Z_p(t) \\ Z_q(t) \end{bmatrix}$, where $Z_p(t) \in \mathcal{M}_{p1}$ and $Z_q(t) \in \mathcal{M}_{q1}$. Taking into account the above expressions, we arrive easily at (3) and (4).

From [1, 7, 27, 48-52] it is known that the subsystem (3) has the general solution

$$Z_p(t) = e^{J_p(t-t_0)} C, \quad (5)$$

where $\sum_{j=1}^v p_j = p$ and $C \in \mathcal{M}_{m1}$ constant.

Proposition 2.1. The subsystem (4) has the unique solution

$$Z_q(t) = 0_{q,q}. \quad (6)$$

Proof. Let q_* be the index of the nilpotent matrix H_q , i.e. $(H_q^{q_*} = 0_{q,q})$, we obtain the following equations

$$\begin{aligned} H_q Z'_q(t) &= Z_q(t), \\ H_q^2 Z'_q(t) &= H_q Z_q(t), \\ H_q^3 Z'_q(t) &= H_q^2 Z_q(t), \\ &\vdots \\ H_q^{q_*} Z'_q(t) &= H_q^{q_*-1} Z_q(t) \end{aligned}$$

and

$$\begin{aligned} H_q Z'_q(t) &= Z_q(t), \\ H_q^2 Z''_q(t) &= H_q Z'_q(t), \\ H_q^3 Z'''_q(t) &= H_q^2 Z''_q(t), \\ &\vdots \\ H_q^{q*} Z^{(q*)}_q(t) &= H_q^{(q*-1)} Z_q(t). \end{aligned}$$

The sum of the above relations gives, (note $H_q^{q*} = 0_{qq}$) is the solution (6).

The boundary value problem

A necessary and sufficient condition for the boundary value problem to be consistent is given by the following result

Theorem 2.1. The boundary value problem (1) is consistent, if and only if

$$D, E \in \text{colspan}[AQ_p] = \text{colspan}[BQ_p e^{J_p(t_N-t_0)}] \quad (7)$$

Where Q_p has column vectors all the linear independent eigenvectors of the finite generalized eigenvalues of $sF - G$.

Proof. Let $Q = [Q_p Q_q]$, where $Q_p \in \mathcal{M}_{mp}$ and $Q_q \in \mathcal{M}_{mq}$. Combining (5), (5), we obtain

$$Y(t) = QZ(t) = [Q_p Q_q] \begin{bmatrix} Q_p e^{J_p(t-t_0)} C \\ 0_{q1} \end{bmatrix}$$

and

$$Y(t) = Q_p e^{J_p(t-t_0)} C.$$

This solution exists if and only if

$$D = AY(t_0)$$

$$E = BY(t_N)$$

and

$$D = AQ_p C$$

$$E = BQ_p e^{J_p(t_N-t_0)} C,$$

or,

$$D, E \in \text{colspan}[AQ_p] = \text{colspan}[BQ_p e^{J_p(t_N-t_0)}].$$

The columns of Q_p are the p eigenvectors of the finite elementary divisors (eigenvalues) of the pencil $sF - G$. (see [9, 25, 35, 40, 44, 48] for algorithms

for the computation of Q_p)

It is obvious that, if there is a solution of the boundary value problem, it needs not to be unique. The necessary and sufficient conditions, for uniqueness, when the problem is consistent, are given by the following theorem.

Theorem 2.2 Assume the boundary value problem (1). Then when it is consistent, it has a unique solutions if and only

$$\text{rank}[AQ_p] = \text{rank}[BQ_p e^{J_p(t_N-t_0)}] = p \quad (8)$$

and the linear system

$$\begin{aligned} AQ_p C &= D \\ BQ_p e^{J_p(t_N-t_0)} C &= E \end{aligned} \quad (9)$$

gives a unique solution for constant column C . Then the unique solution is given by

$$Y(t) = Q_p e^{J_p(t-t_0)} C. \quad (10)$$

Proof Let the boundary value problem (1) be consistent, then from Theorem 2.1 the solution is

$$Y(t) = Q_p e^{J_p(t-t_0)} C$$

with

$$D = AY(t_0)$$

$$E = BY(t_N)$$

and

$$D = AQ_p C$$

$$E = BQ_p e^{J_p(t_N-t_0)} C.$$

It is clear that for given A, B, D, E the problem (1) has a unique solution if and only if the system (9) has a unique solution. Since $AQ_p, BQ_p e^{J_p(t-t_0)} \in \mathcal{M}_{mp}$, the solution is unique for the system (9) if and only if the matrices $AQ_p, BQ_p e^{J_p(t-t_0)}$ are left invertible ($\text{rank}[AQ_p] = \text{rank}[BQ_p e^{J_p(t-t_0)}] = p$) and both equations give the same unique solution for the constant column C .

3 Numerical Example

Consider the boundary value problem (1). Where

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, G = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -4 & 2 & 2 & -3 & -2 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 \end{bmatrix}.$$

and A the identity matrix and B a matrix that satisfies the equation $BQ_p e^{J_p(t_N-t_0)} = Q_p$. The invariants of $sF - G$ are $s - 1$, $s - 2$, $s - 3$ (finite elementary divisors) and $\bar{s}^3$ (infinte elementary divisor of degree 3). Then

$$J_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

and the columns of Q_p are the eigenvectors of the generalized eigenvalues 1, 2, 3 respectively. Then

$$AQ_p = BQ_p e^{J_p(t_N-t_0)} = \begin{bmatrix} 3 & -5 & 3 & -5 & 3 & -5 \\ 1 & -1 & 2 & -2 & 4 & -4 \\ 1 & -1 & 3 & -3 & 9 & -9 \end{bmatrix}^T. \quad (11)$$

Example 1

Let

$$D = E = [1 \quad -3 \quad -2 \quad 0 \quad -10 \quad 8]^T.$$

Then

$$D, E \in \text{colspan}[AQ_p] = \text{colspan}[BQ_p e^{J_p(t_N-t_0)}]$$

and by calculating C from (9) we get

$$C = [1 \quad -1 \quad -1]^T$$

and the unique solution of the system by substituting in (10) is

$$Y(t) = \begin{bmatrix} 3e^t - e^{2t} - e^{3t} \\ -5e^t + e^{2t} + e^{3t} \\ 3e^t - 2e^{2t} - 3e^{3t} \\ -5e^t + 2e^{2t} + 3e^{3t} \\ 3e^t - 4e^{2t} - 9e^{3t} \\ -5e^t + 4e^{2t} + 9e^{3t} \end{bmatrix}.$$

Example 2

Let

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}^T.$$

Then

$$D \notin \text{colspan}[AQ_p] = \text{colspan}[BQ_p e^{J_p(t_N - t_0)}]$$

and thus the boundary value problem is not consistent.

Acknowledgement

The authors would like to express their sincere gratitude to Professor G.I. Kalogeropoulos for his helpful and fruitful discussions that led to necessary changes and modifications in the proof of the theorems. Moreover, we are very grateful to the anonymous referees for their valuable suggestions that clearly improved this article.

Conclusions

The aim of this article was to give necessary and sufficient conditions for existence and uniqueness of solutions for generalized linear discrete time boundary value problems of a class of linear rectangular matrix differential equations whose coefficients are square constant matrices. By taking into consideration that the relevant pencil is regular, we get effected by the Weierstrass canonical form in order to decompose the differential system into two sub-systems. Afterward, we provide analytic formulas when we have a consistent problem. Moreover, as a further extension of the present paper, we can discuss the case where the pencil is singular. Thus, the Kronecker canonical form is required. For all these, there is some research in progress.

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Received: October 15, 2014; Published: January 7, 2015