**Applied Mathematical Sciences, Vol. 9, 2015, no. 11, 507 - 514**
**HIKARI Ltd, www.m-hikari.com**
**http://dx.doi.org/10.12988/ams.2015.411941**

# 1-Movable Connected Dominating Sets in Graphs

**Jocecar Lomarda**[1]

College of Teacher Education
Bohol Island State University-Main Campus
CPG North Avenue, 6300 Tagbilaran City, Bohol, Philippines

**Sergio R. Canoy, Jr.**

Department of Mathematics and Statistics
Mindanao State University-Iligan Institute of Technology
Tibanga Highway, 9200 Iligan City, Philippines



**Abstract**

A connected dominating set $C$ in a connected nontrivial graph $G$ is a *1-movable connected dominating set* in $G$ if for every $v \in C$, either $C \setminus \{v\}$ is a connected dominating set, or there exists a vertex $u \in (V(G) \setminus C) \cap N(v)$ such that $(C \setminus \{v\}) \cup \{u\}$ is a connected dominating set of $G$. The minimum cardinality of a 1-movable connected dominating set of $G$, denoted by $\gamma_{mc}^1(G)$ is the *1-movable connected domination number* of $G$. A 1-movable connected dominating set with cardinality $\gamma_{mc}^1(G)$ is called a *minimum 1-movable connected dominating set* or a $\gamma_{mc}^1$-set of $G$. In this paper, we characterize those graphs $G$ having a 1-movable connected dominating set. We also characterize the 1-movable connected dominating sets in the join of graphs and determine the corresponding 1-movable connected domination number of these graphs.

**Mathematics Subject Classification:** 05C69

[1]This research is partially funded by the Commission on Higher Education, Philippines under Faculty Development Program Phase II

**Keywords:** connected domination, 1-movable connected domination, external and internal private neighbors, join

# 1 Introduction

Let $G = (V(G), E(G))$ be a graph with $n = |V(G)|$ and $m = |E(G)|$. For any vertex $v \in V(G)$, the *open neighborhood* of $v$ is the set $N_G(v) = N(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of $v$ is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. If $S \subseteq V(G)$, then the *open neighborhood* of $S$ is the set $N_G(S) = N(S) = \cup_{v \in S} N_G(v)$ and the *closed neighborhood* of $S$ is the set $N_G[S] = N[S] = S \cup N(S)$. A set $S \subseteq V(G)$ is a *dominating set* of $G$ if for every $v \in V(G) \backslash S$, there exists $u \in S$ such that $uv \in E(G)$, that is, $N_G[S] = V(G)$. The *domination number* of $G$, denoted by $\gamma(G)$ is the smallest cardinality of a dominating set of $G$. A dominating set of $G$ with cardinality equal to $\gamma(G)$ is called a $\gamma$-set of $G$. Now, if $S$ is a dominating set of $G$, then a vertex $u$ is a *private neighbor* of $v \in S$ if $N(u) \cap S = \{v\}$. If $u \in S$, then $u$ is an *internal private neighbor* of $v \in S$, otherwise, $u$ is an *external private neighbor* of $v \in S$. The set of internal private neighbors of $v \in S$ with respect to $S$ is denoted by $ipn\,(v; S)$ and the set of external private neighbors of $v \in S$ with respect to $S$ is denoted by $epn\,(v; S)$.

A dominating set $S \subseteq V(G)$ is called a *connected dominating set* of $G$ if the subgraph $\langle S \rangle$ induced by $S$ is connected. The *connected domination number* of $G$, denoted by $\gamma_c(G)$ is the smallest cardinality of a connected dominating set of $G$. A connected dominating set $S$ of $G$ with $|S| = \gamma_c(G)$ is called a $\gamma_c$-set. A connected dominating set $C$ in $G$ is a *1-movable connected dominating set* of $G$ if for every $v \in C$, either $C \setminus \{v\}$ is a connected dominating set, or there exists a vertex $u \in (V(G) \setminus C) \cap N(v)$ such that $(C \setminus \{v\}) \cup \{u\}$ is a connected dominating set of $G$. The minimum cardinality of a 1-movable connected dominating set of $G$, denoted by $\gamma_{mc}^1(G)$ is the *1-movable connected domination number* of $G$. A 1-movable connected dominating set with cardinality $\gamma_{mc}^1(G)$ is called a *minimum 1-movable connected dominating set* or a $\gamma_{mc}^1$-set of $G$. Moreover, 1-movable domination and 1-movable total domination in graphs are introduced and investigated in [1], [2], and [3].

# 2 Results

**Remark 2.1** *Every connected dominating set contains every cut-vertex.*

The next result characterizes all connected nontrivial graphs having a 1-movable connected dominating set.

**Theorem 2.2** *A connected nontrivial graph $G$ has a 1-movable connected dominating set if and only if $G$ has no cut-vertices.*

**Proof.** Suppose that $G$ has a 1-movable connected dominating set, say $S$. Suppose further that $G$ has a cut-vertex $v$. Then, by the remark 2.1, $v \in S$. Hence, $S \setminus \{v\}$ and $(S \setminus \{v\}) \cup \{u\}$, where $u \in V(G) \setminus S$ are not connected dominating sets of G. This implies that $S$ is not a 1-movable connected dominating set, contrary to our assumption. Thus, $G$ has no cut-vertices.

Conversely, suppose that $G$ has no cut-vertices. Let $S = V(G)$. Then, clearly, $S$ is a connected dominating set. Let $v \in S$. Since $G$ has no cut-vertices, $S \setminus \{v\}$ is a connected dominating set of $G$. Hence, $S$ is a 1-movable connected dominating set of $G$. $\square$

**Remark 2.3** *For any connected nontrivial graph $G$ without cut-vertices, $\gamma_c(G) \leq \gamma^1_{mc}(G)$.*

**Remark 2.4** *Let $G$ be a connected nontrivial graph without cut-vertices. Then $1 \leq \gamma^1_{mc}(G) \leq n$, where $n = |V(G)|$, and these bounds are sharp.*

To see this, consider $G_1 = C_4$ and $G_2 = K_5$. It can be verified that $\gamma^1_{mc}(G_1) = \gamma^1_{mc}(C_4) = 4$ and $\gamma^1_{mc}(G_2) = \gamma^1_{mc}(K_5) = 1$.

The next result says that all nontrivial complete graphs attain the lower bound of the inequality in Remark 2.4.

**Lemma 2.5** $\gamma^1_{mc}(K_n) = 1$ *for all $n \geq 2$.*

**Proof.** Choose any $x \in V(K_n)$ and let $S = \{x\}$. Then $S$ is a connected dominating set of $K_n$. If $y \in V(K_n) \setminus \{x\}$, *then* $(S \setminus \{x\}) \cup \{y\} = \{y\}$ is a connected dominating set of $G$. Thus, $S$ is a 1-movable connected dominating set of $K_n$. Therefore, by Remark 2.4, $\gamma^1_{mc}(K_n) = 1$. $\square$

**Theorem 2.6** *Let $G$ be a connected nontrivial graph without cut-vertices. Then $\gamma^1_{mc}(G) = 1$ if and only if $G = K_2$ or $G \cong K_2 + H$ for some graph $H$.*

**Proof.** Suppose that $\gamma^1_{mc}(G) = 1$. If $|V(G)| = 2$, then $G = K_2$. Suppose that $|V(G)| > 2$. Then $G$ has a $\gamma^1_{mc}$-set say, $S = \{x\}$ for some $x \in V(G)$. Since $x$ dominates $G$, it follows that $V(G) \setminus \{x\} \subseteq N(x)$. Since S is a 1-movable connected dominating set of $G$, there exists $y \in (V(G) \setminus S) \cap N(x)$ such that $(S \setminus \{x\}) \cup \{y\} = \{y\}$ is a connected dominating set of $G$. Hence, $V(G) \setminus \{y\} \subseteq N(y)$. Thus, $xy \in E(G)$. Let $H = \langle V(G) \setminus \{x, y\} \rangle$. Then, $G = \langle \{x, y\} \rangle + H \cong K_2 + H$.

Conversely, if $G = K_2$, then by Lemma 2.5, $\gamma^1_{mc}(G) = \gamma^1_{mc}(K_2) = 1$. Suppose that $G \cong K_2 + H$ for some graph $H$. Let $V(K_2) = \{a, b\}$ and set $S = \{a\}$. Then $S$ is a connected dominating set of $G$ and $S \setminus \{a\} \cup \{b\} = \{b\}$ is a connected dominating set of $G$. Thus $S$ is a $\gamma^1_{mc}$-set of $G$. Thus, $\gamma^1_{mc}(G) = |S| = 1$ . $\square$

**Theorem 2.7** *Let $G$ be a connected graph of order $n \geq 3$ having no cut-vertices. Then $\gamma^1_{mc}(G) = 2$ if and only if the following conditions hold:*

(*i*) *$G \ncong K_2 + H$ for any graph $H$; and*

(*ii*) *there exist adjacent vertices $x$ and $y$ that dominate $G$ such that*

(*a*) *$epn(x; \{x, y\}) \subseteq N_G(z)$ for some $z \in N_G(x) \cap N_G(y)$ and*

(*b*) *$epn(y; \{x, y\}) \subseteq N_G(w)$ for some $w \in N_G(x) \cap N_G(y)$.*

**Proof.** Suppose that $\gamma^1_{mc}(G) = 2$. Then by Theorem 2.6, (*i*) holds. Let $S = \{x, y\}$ be a $\gamma^1_{mc}$-set of $G$. Since $S$ is a connected dominating set of $G$, $xy \in E(G)$. Also, since $S$ is a 1-movable connected dominating set of $G$, there exists $z \in N_G(x)$ such that $(S \setminus \{x\}) \cup \{z\} = \{y, z\}$ is a connected dominating set of $G$. Hence, $yz \in E(G)$, that is, $z \in N_G(x) \cap N_G(y)$. Let $v \in epn(x, S)$. Since $vy \notin E(G)$ and $\{z, y\}$ is a dominating set of $G$, it follows that $v \in N_G(z)$. Since $v$ was arbitrarily chosen, $epn\,(x; \{x, y\}) \subseteq N_G(z)$. Similarly, (*b*) holds.Thus, (*ii*) holds.

Conversely, suppose that (*i*) and (*ii*) holds. Then, by Theorem 2.6, $\gamma^1_{mc}(G) \geq 2$. Let $S = \{x, y\}$ where $x$ and $y$ satisfy (*ii*). Then $S$ is a connected dominating set of $G$. Moreover, by (*a*), $(S \setminus \{x\}) \cup \{z\} = \{y, z\}$ is a connected dominating set of $G$. Similarly, by (*b*), $(S \setminus \{y\}) \cup \{w\} = \{x, w\}$ is a connected dominating set of $G$. Thus, $S$ is a 1-movable connected dominating set of $G$ and so a $\gamma^1_{mc}$-set of $G$. Thus, $\gamma^1_{mc}(G) = |S| = 2$. □

The next result characterizes the concept of 1-movable connected dominating set in terms of the concept of private neighbors.

**Theorem 2.8** *Let $G$ be a connected graph without cut-vertices. A subset $S$ of $V(G)$ is a 1-movable connected dominating set of $G$ if and only if $S$ is a connected dominating set of $G$ and for each $v \in S$, either $epn(v; S) = ipn(v; S) = \varnothing$ or there exists $u \in (V(G) \setminus S) \cap N(v)$ such that $epn(v; S) \cup ipn(v; S) \subseteq N[u]$.*

**Proof.** Suppose that $S$ is a 1-movable connected dominating set of $G$. Then $S$ is a connected dominating set of $G$. Let $v \in S$. If $S \setminus \{v\}$ is a connected dominating set of $G$, then every vertex $w$ in $(V(G) \setminus S) \cap N(v)$ is adjacent to some vertex in $S \setminus \{v\}$. This implies that $epn(v; S) = \varnothing$. Also since $\langle S \setminus \{v\} \rangle$ is connected, $ipn(v; S) = \varnothing$. Suppose that $S \setminus \{v\}$ is not a connected dominating set of $G$. Then, by assumption, there exists a vertex $u \in (V(G) \setminus S) \cap N(v)$ such that $S_v = (S \setminus \{v\}) \cup \{u\}$ is a connected dominating set of $G$. Let $z \in epn(v; S)$. Then $z \in N[u]$ since $S_v$ is a dominating set of $G$. Thus, $epn(v; S) \subseteq N[u]$. Also, if $y \in ipn(v; S)$, then $y \in N(u)$ since $\langle S_v \rangle$ is connected. Thus, $epn(v; S) \cup ipn(v; S) \subseteq N[u]$.

For the converse, suppose that $S$ is a connected dominating set satisfying the given condition. Let $v \in S$. If $epn(v;S) = ipn(v;S) = \varnothing$, then $S \setminus \{v\}$ is a connected dominating set of $G$. Suppose that there exists $u \in (V(G) \setminus S) \cap N(v)$ such that $epn(v;S) \cup ipn(v;S) \subseteq N[u]$. Set $S_v = (S \setminus \{v\}) \cup \{u\}$ and let $x \in V(G) \setminus S_v$. If $x = v$ or $x \in epn(v;S)$, then $xu \in E(G)$. If $x \notin \{v\} \cup epn(v;S)$, then $xy \in E(G)$ for some $y \in S \setminus \{v\}$ since $S$ is a dominating set of $G$. Moreover, since $ipn(v;S) \subseteq N(u)$, $\langle S_v \rangle$ is connected. Thus, $S_v$ is a connected dominating set of $G$. This shows that $S$ is a 1-movable connected dominating set of $G$. □

The next result characterizes the 1-movable connected dominating sets in the join of two connected nontrivial graphs.

**Theorem 2.9** *Let $G$ and $H$ be connected nontrivial graphs. Then $S \subseteq V(G+H)$ is a 1-movable connected dominating set of $G+H$ if and only if one of the following statements holds:*

($i$) *$S$ is a connected dominating set of $G$ such that if $|S| = 1$, then either $S$ is a 1-movable connected dominating set of $G$ or there exists $u \in V(H)$ such that $\{u\}$ is a (connected) dominating set in $H$.*

($ii$) *$S$ is a connected dominating set of $H$ such that if $|S| = 1$, then either $S$ is a 1-movable connected dominating set of $H$ or there exists $v \in V(G)$ such that $\{v\}$ is a (connected) dominating set in $G$.*

($iii$) *$S \cap V(G) \neq \varnothing$ and $S \cap V(H) \neq \varnothing$.*

**Proof.** Let $S \subseteq V(G+H)$ be a 1-movable connected dominating set of $G+H$. If $S \cap V(G) \neq \varnothing$ and $S \cap V(H) \neq \varnothing$, then ($iii$) holds. Suppose that $S \cap V(G) = \varnothing$ or $S \cap V(H) = \varnothing$. Then $S \subseteq V(G)$ or $S \subseteq V(H)$. Suppose that $S \subseteq V(G)$. Since $S$ is a connected dominating set of $G+H$, it follows that $S$ is also a connected dominating set of $G$. Suppose that $|S| = 1$, say $S = \{v\}$ for some $v \in V(G)$. Since $S$ is a 1-movable connected dominating set of $G+H$, there exists $u \in (V(G+H) \setminus S) \cap N(v)$ such that $(S \setminus \{v\}) \cup \{u\} = \{u\}$ is a connected dominating set of $G+H$. If $u \in V(G)$, then $(S \setminus \{v\}) \cup \{u\} = \{u\}$ is a connected dominating set of $G$. Hence, $S$ is a 1-movable connected dominating set of $G$. If $u \in V(H)$, then $(S \setminus \{v\}) \cup \{u\} = \{u\}$ is a connected dominating set of $H$. Thus, ($i$) holds. Similarly, if $S \subseteq V(H)$, then ($ii$) holds.

For the converse, suppose that ($i$) holds. Then, by definition of $G+H$, $S$ is a connected dominating set of $G+H$. Suppose that $|S| \geq 2$. Let $v \in S$ and choose any $u \in V(H)$. Then $(S \setminus \{v\}) \cup \{u\}$ is a connected dominating set of $G+H$. Since $v$ is arbitrary, $S$ is a 1-movable connected dominating set of $G+H$. Suppose that $|S| = 1$. Then $S = \{v\}$ for some $v \in V(G)$. Suppose $S$ is a 1-movable connected dominating set of $G$. Then there exists

$u \in (V(G) \setminus S) \cap N(v)$ such that $(S \setminus \{v\}) \cup \{u\} = \{u\}$ is a connected dominating set of $G$ (and hence of $G + H$). If $S_1 = \{w\}$ is a $\gamma_c$-set for some $w \in V(H)$, then $w \in V(H) \cap N(v)$ and $(S \setminus \{v\}) \cup \{w\} = \{w\}$ is a connected dominating set of $H$ (hence of $G + H$). So in either case, $S$ is a 1-movable connected dominating set of $G+H$. Similarly, if $(ii)$ holds, then $S$ is a 1-movable connected dominating set of $G + H$. Suppose $(iii)$ holds. Then clearly, $S$ is a connected dominating set of $G + H$. Let $S_1 = S \cap V(G) \neq \varnothing$ and $S_2 = S \cap V(H) \neq \varnothing$. Then $S = S_1 \cup S_2$. Let $v \in S$. Suppose that $v \in S_1$. If $|S_1| = 1$, then there exists $u \in (V(G) \setminus S_1) \cap N(v)$ (since $G$ is a nontrivial connected graph) such that $(S \setminus \{v\}) \cup \{u\}$ is a connected dominating set of $G + H$. If $|S_1| \geq 2$, then $S_1 \setminus \{v\} \neq \varnothing$. Hence, in this case, it follows that $S \setminus \{v\}$ is a connected dominating set. Similar arguments can be used to come up with the desired property of $S$ if $v \in S_2$. Therefore, $S$ is a 1-movable connected dominating set of $G + H$. □

**Corollary 2.10** *Let $G$ and $H$ be connected nontrivial graphs. Then*

$$\gamma_{mc}^1(G + H) = \begin{cases} 1, & \text{if } \gamma_c(G) = 1 = \gamma_c(H) \text{ or } \gamma_{mc}^1(G) = 1 \text{ or } \gamma_{mc}^1(H) = 1 \\ 2, & \text{otherwise.} \end{cases}$$

**Theorem 2.11** *Let $H$ be a connected nontrivial graph. Then $S \subseteq V(K_1 + H)$ is a 1-movable connected dominating set of $K_1 + H$ if and only if one of the following statements holds:*

- $(i)$ *$S = V(K_1)$ and there exists $u \in V(H)$ such that $\{u\}$ is a (connected) dominating set in $H$.*
- $(ii)$ *$S = V(K_1) \cup S_1$, where $\varnothing \neq S_1 \subseteq V(H)$ and either*
  - $(a)$ *$S_1$ is a connected dominating set of $H$ or*
  - $(b)$ *$S_1 \cup \{c\}$ is a connected dominating set of $H$ for some $c \in V(H) \setminus S_1$.*
- $(iii)$ *$S$ is a connected dominating set of $H$.*

**Proof.**

Let $V(K_1) = \{z\}$. Suppose that $S$ is a 1-movable connected dominating set of $K_1 + H$. Consider the following cases:

Case 1: $z \in S$

Suppose that $S = \{z\}$. Since $S$ is a 1-movable connected dominating set of $K_1 + H$, there exists $u \in V(H) \cap N(z)$ such that $(S \setminus \{z\}) \cup \{u\} = \{u\}$ is a connected dominating set of $G + H$ (and hence of $H$). Thus, statement $(i)$ holds. Next, suppose that $S = \{z\} \cup S_1$ where $\varnothing \neq S_1 \subseteq V(H)$. Since $S$ is a 1-movable connected dominating set of $K_1 + H$, either $S \setminus \{z\} = S_1$ is a connected dominating set in $K_1 + H$ (also in $H$) or there exists $c \in V(H) \setminus S_1$ such that $S_1 \cup \{c\}$ is a connected dominating set of $K_1 + H$ (also in $H$). Therefore statement $(ii)$ holds.

Case 2: $z \notin S$

If $z \notin S$, then $S \subseteq V(H)$. Since $S$ is a connected dominating set of $K_1 + H$, $S$ is also a connected dominating set of $H$. Hence, statement $(iii)$ holds.

For the converse, suppose that $(i)$ holds. Then $S$ is a connected dominating set of $K_1 + H$ and $H$ has a $\gamma_c$-set say $S_1 = \{w\}$ for some $w \in V(H)$. Thus, $(S \setminus \{z\}) \cup \{w\} = \{w\}$ is a connected dominating set of $K_1 + H$. Hence, $S$ is a 1-movable connected dominating set of $K_1 + H$. Suppose that $(ii)$ holds. Then $S$ is a connected dominating set of $K_1 + H$. Let $v \in S$. Suppose that $v = z$. If $(a)$ holds, then $S \setminus \{v\} = S_1$ is a connected dominating set of $H$ (and hence of $K_1 + H$). If $(b)$ holds, then $(S \setminus \{v\}) \cup \{c\} = S_1 \cup \{c\}$ is a connected dominating set of $H$ (and hence of $K_1 + H$). Next, suppose that $v \in S_1$. Since $z \in S \setminus \{v\}$, $S \setminus \{v\}$ is a connected dominating set of $K_1 + H$. Hence, in either case, $S$ is a 1-movable connected dominating set of $K_1 + H$. Finally, suppose that $(iii)$ holds. Then clearly, $S$ is a connected dominating set of $K_1 + H$. Let $v \in S$. Then $S \setminus \{v\}) \cup \{z\}$ is a connected dominating set of $K_1 + H$. Hence, $S$ is a 1-movable connected dominating set of $K_1 + H$. □

**Corollary 2.12** *Let $H$ be a connected graph of order $n \geq 2$. Then*

$$\gamma_{mc}^1 (K_1 + H) = \gamma_c (H).$$

**Theorem 2.13** *Let $m \geq 2$ and $n \geq 2$ be positive integers. Then $S \subseteq V(K_{m,n})$ is a 1-movable connected dominating set of $K_{m,n} = \overline{K_m} + \overline{K_n}$ if and only if $|S \cap V(\overline{K_m})| \geq 2$ and $|S \cap V(\overline{K_n})| \geq 2$.*

**Proof.** Let $S_1 = S \cap V(\overline{K_m})$ and $S_2 = S \cap V(\overline{K_n})$. Suppose that $S$ is a 1-movable connected dominating set of $K_{m,n}$. Since $\langle S \rangle$ is connected, $S_1 \neq \varnothing$ and $S_2 \neq \varnothing$. Suppose that $|S_1| = 1$, say $S_1 = \{v\}$. Then $\langle S \setminus \{v\} \rangle = \langle S_2 \rangle$ is not connected. Also, $\langle (S \cup \{v\}) \cup \{x\} \rangle = \langle S_2 \setminus \{x\} \rangle$ is not connected for all $x \in V(H) \setminus S_2$. Thus, $S$ is not a 1-movable connected dominating set of $K_{m,n}$, contrary to our assumption. Therefore, $|S_1| \geq 2$. Similarly, $|S_2| \geq 2$.

Conversely, suppose that $|S \cap V(\overline{K_m})| \geq 2$ and $|S \cap V(\overline{K_n})| \geq 2$. Then $S = S_1 \cup S_2$ is a connected dominating set of $K_{m,n}$. Let $v \in S$. If $v \in S_1$, then $S_1 \setminus \{v\} \neq \varnothing$. Hence, $S \setminus \{v\}$ is a connected dominating set of $K_{m,n}$. Also, if $v \in S_2$, then $S_2 \setminus \{v\} \neq \varnothing$. Hence, $S \setminus \{v\}$ is a connected dominating set of $K_{m,n}$. Therefore $S$ is a 1-movable connected dominating set of $K_{m,n}$. □

**Corollary 2.14** *Let $m \geq 2$ and $n \geq 2$ be positive integers. Then*

$$\gamma_{mc}^1 (K_{m,n}) = 4.$$

# References

[1] J. Blair, R. Gera and S. Horton. *Movable dominating sensor sets in networks*. Journal of Combinatorial Mathematics and Combinatorial Computing, 77 (2011), 103-123.

[2] R. G. Hinampas, Jr., and S. R. Canoy. *1-movable domination in graphs*. Applied Mathematical Sciences, 8(2014), no. 172, 8565-8571. http://dx.doi.org/10.12988/ams.2014.410859

[3] J. Lomarda, and S. R. Canoy. *1-movable total dominating sets in graphs*. International Journal of Mathematical Analysis, 8(2014), no. 55, 2703-2709. http://dx.doi.org/10.12988/ijma.2014.410326

**Received: November 17, 2014; Published: January 7, 2015**