Graceful Labeling of Roman Rings Having Cycle with 6 Vertices

G. Sathiamoorthy

School of Humanities and Sciences, SASTRA University, India

T. N. Janakiraman

Department of Mathematics, NIT Trichy, India

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Abstract

Roman rings can be obtained by introducing n copies of cycle C_6 with 6 vertices, which are merged respectively to n teeth of comb graph $P_n \odot L_1$. In this paper, it is proved Roman rings with cycle C_6 are graceful.

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1 Introduction

Graphs considered in this paper are simple finite and undirected. In general $G(V, E)$ denotes the graph G with vertex set $V(G)$, edge set $E(G)$, such that $|V(G)| = p$ vertices $|E(G)| = q$ edges. A labeling of the vertices of G with the numbers from 0 to q is an injective map $\phi : V \rightarrow \{0, 1, \dots, q\}$. A graph G is graceful if there exists a labeling of its vertices such that the map $\phi^* : E \rightarrow \{1, 2, \dots, q\}$ given by $\phi^*(uv) = |\phi(u) - \phi(v)|$, where $u, v \in V$ and $uv \in E$ is a bijection.

A graph that admits graceful labeling is called graceful graph. The notation

graceful labeling was introduced Rosa[3] with the name valuation. Gallian[2] gives the extensive survey of contributions to graceful labeling of variety of graphs. Rosa[3], [4] showed that the cycle C_n is graceful if and only if $n \equiv 0$ or $3 \pmod{4}$. Bhat-Nayak and Selvam [1] have shown that the n -cone (also called the n -point suspension of C_m) $C_m + \overline{K_n}$ is graceful when $m \equiv 0$ or $3 \pmod{12}$. They also proved the gracefulness of $C_4 + \overline{K_n}$, $C_5 + \overline{K_2}$, $C_7 + \overline{K_n}$, $C_9 + \overline{K_2}$, $C_{11} + \overline{K_n}$ and $C_{19} + \overline{K_n}$. Seo [5] proved Gracefulness of the union of cycles and paths.

2 Main Result

Let $R_1, R_2, \dots, R_n$ be n copies of cycle C_6 (we term here as rings). Let the supporting points on the n rings $R_1, R_2, \dots, R_n$ be $t_1, t_2, \dots, t_n$, which are merged respectively to n teeth of comb graph $P_n \odot L_1$. Let $b_1, b_2, b_3, \dots, b_n$ be base points of the comb graph from which n rings of equal length say m , are hanging, each of which at a tooth of n teeth respectively. The resulting structure is called Roman rings $R(6, n)$. Let the points of i^{th} ring be $c_1^i, c_2^i, \dots, c_{m-1}^i$ for $i = 1, 2, \dots, n$.

From the above definition of $R(6, n)$ graph, $|V(R(6, n))| = 7n$. Also, the number of edges of $R(6, n)$ is $|E(R(6, n))| = 8n - 1$.

Theorem 2.1. *The Roman rings $R(6, n)$ is graceful.*

Figure 1: General form of $R(6, n)$

The labeling of $(n - 1)$ rings are as follows:-

Step 1 : $\phi(b_1) = 0, \phi(t_1) = q, \phi(c_1^1) = 1, \phi(c_2^1) = 2$.

Step 2 : $\phi(b_2) = q - 3, \phi(t_2) = 3, \phi(c_1^2) = q - 1, \phi(c_2^2) = q - 2$.

(Let $n = 2\lambda + 1$ for odd and $n = 2\lambda$ for even)

Case 1: n is odd, then

Step 3 : $\phi(b_{2i+1}) = \phi(b_{2i-1}) + 4, 1 \leq i \leq \lambda - 1.$

Step 4 : $\phi(t_{2i+1}) = q - \phi(b_{2i+1}), 1 \leq i \leq \lambda - 1.$

Step 5 : $\phi(b_{2i+2}) = \phi(b_{2i}) - 4, 1 \leq i \leq \lambda - 1.$

Step 6 : $\phi(t_{2i+2}) = \phi(t_{2i}) + 4, 1 \leq i \leq \lambda - 1.$

Case 2: n is even, then

Step 3 : $\phi(b_{2i+1}) = \phi(b_{2i-1}) + 4, 1 \leq i \leq \lambda - 1.$

Step 4 : $\phi(t_{2i+1}) = q - \phi(b_{2i+1}), 1 \leq i \leq \lambda - 1.$

Step 5 : $\phi(b_{2i+2}) = \phi(b_{2i}) - 4, 1 \leq i \leq \lambda - 1.$

Step 6 : $\phi(t_{2i+2}) = \phi(t_{2i}) + 4, 1 \leq i \leq \lambda - 1.$

Step 7 : $\phi(c_3^1) = (n - 2) 2 + 12, \phi(c_4^1) = \phi(c_3^1) + 2.$

Step 8 : $\phi(c_3^2) = (n - 3) 6 + 7, \phi(c_4^2) = \phi(c_3^2) + 2.$

Case 1: n is odd (continue), then

Step 9 : $\phi(c_3^{2i+1}) = \phi(c_3^{2i-1}) + 8, 1 \leq i \leq \lambda - 1.$

Step 10 : $\phi(c_4^{2i+1}) = \phi(c_4^{2i-1}) + 8, 1 \leq i \leq \lambda - 1.$

Step 11 : $\phi(c_3^{2i+2}) = \phi(c_3^{2i}) - 8, 1 \leq i \leq \lambda - 1.$

Step 12 : $\phi(c_4^{2i+2}) = \phi(c_4^{2i}) - 8, 1 \leq i \leq \lambda - 1.$

Case 2: n is even(continue), then

Step 9 : $\phi(c_3^{2i+1}) = \phi(c_3^{2i-1}) + 8, 1 \leq i \leq \lambda - 1.$

Step 10 : $\phi(c_4^{2i+1}) = \phi(c_4^{2i-1}) + 8, 1 \leq i \leq \lambda - 1.$

Step 11 : $\phi(c_3^{2i+2}) = \phi(c_3^{2i}) - 8, 1 \leq i \leq \lambda - 1.$

Step 12 : $\phi(c_4^{2i+2}) = \phi(c_4^{2i}) - 8, 1 \leq i \leq \lambda - 1.$

The remaining $(n - 1) 2 + 7$ edges starting from 1 to $2(n - 3) + 12$. Let $e_i = 2(n - 3) + 12$. The remaining edges formed in n pair relations starting from $(e_i, e_i - 2), (e_i - 1, e_i - 3), (e_i - 4, e_i - 6), (e_i - 5, e_i - 7), \dots$. The labeling of $\phi(c_5)$ are as follows :-

Step 13 : select maximum edge value pair among $(\phi(c_3^i), \phi(c_4^i)), i = 1$ to $n - 1.$

Step 14 : substitute maximum edge value pair and the resultant vertex lies in $3(n - 2) + 1$ to $q - 2(n - 1).$

Step 15 : if not continue with other decreasing order of vertex pairs.

Step 16 : select 2nd edge pair and continue the above steps.

The remaining 7 edges consists of $(n - 1)^{th}$ pair in the above n pairs and remaining edges from 1 to 7.

Last ring consists of 6 edges.

Now, induced edge labeling are as follows :-

$$\mathbf{1} : \phi^*(b_1t_1) = q. \quad \mathbf{2} : \phi^*(t_1c_1^1) = q - 1. \quad \mathbf{3} : \phi^*(t_1c_2^1) = q - 2.$$

$$\mathbf{4} : \phi^*(b_1b_2) = q - 3. \quad \mathbf{5} : \phi^*(t_2c_1^2) = q - 4. \quad \mathbf{6} : \phi^*(t_2c_2^2) = q - 5.$$

$$\mathbf{7} : \phi^*(b_2t_2) = q - 6. \quad \mathbf{8} : \phi^*(b_2b_3) = q - 7.$$

Case 1: n is odd, then

Induced edge labeling of odd segment are as follows :-

$$\mathbf{9} : \phi^*(b_{2d+1}t_{2d+1}) = q - 8d, 1 \leq d \leq \lambda - 1.$$

$$\mathbf{10} : \phi^*(t_{2d+1}c_1^{2d+1}) = q - 1 - 8d, 1 \leq d \leq \lambda - 1.$$

$$\mathbf{11} : \phi^*(t_{2d+1}c_2^{2d+1}) = q - 2 - 8d, 1 \leq d \leq \lambda - 1.$$

$$\mathbf{12} : \phi^*(b_{2d+1}b_{2d+2}) = q - 3 - 8d, 1 \leq d \leq \lambda - 1.$$

Induced edge labeling of even segment are as follows :-

$$\mathbf{13} : \phi^*(t_{2d+2}c_1^{2d+2}) = q - 4 - 8d, 1 \leq d \leq \lambda - 1.$$

$$\mathbf{14} : \phi^*(t_{2d+2}c_2^{2d+2}) = q - 5 - 8d, 1 \leq d \leq \lambda - 1.$$

$$\mathbf{15} : \phi^*(b_{2d+1}t_{2d+2}) = q - 6 - 8d, 1 \leq d \leq \lambda - 1.$$

$$\mathbf{16} : \phi^*(b_{2d+2}b_{2d+3}) = q - 7 - 8d, 1 \leq d \leq \lambda - 1.$$

Case 2: n is even, then

Induced edge labeling of odd segment are as follows :-

$$\mathbf{9} : \phi^*(b_{2d+1}t_{2d+1}) = q - 8d, 1 \leq d \leq \lambda - 1.$$

$$\mathbf{10} : \phi^*(t_{2d+1}c_1^{2d+1}) = q - 1 - 8d, 1 \leq d \leq \lambda - 1.$$

$$\mathbf{11} : \phi^*(t_{2d+1}c_2^{2d+1}) = q - 2 - 8d, 1 \leq d \leq \lambda - 1.$$

$$\mathbf{12} : \phi^*(b_{2d+1}b_{2d+2}) = q - 3 - 8d, 1 \leq d \leq \lambda - 2.$$

Induced edge labeling of even segment are as follows :-

$$\mathbf{13} : \phi^*(t_{2d+2}c_1^{2d+2}) = q - 4 - 8d, 1 \leq d \leq \lambda - 2.$$

$$\mathbf{14} : \phi^*(t_{2d+2}c_2^{2d+2}) = q - 5 - 8d, 1 \leq d \leq \lambda - 2.$$

15 : $\phi^*(b_{2d+1}t_{2d+2}) = q - 6 - 8d, 1 \leq d \leq \lambda - 2.$

16 : $\phi^*(b_{2d+2}b_{2d+3}) = q - 7 - 8d, 1 \leq d \leq \lambda - 2.$

Step a : select maximum edge value from the available assigned values of pair adjacent vertices $((c_3^i), (c_4^i)), i = 1$ to $n - 1.$

Step b : substitute maximum edge value pair and the resultant vertex value lies in the vertex set from $3(n - 2) + 1$ to $q - 2(n - 1).$

Step c : if not continue with other decreasing order of vertex pairs.

Step d : select 2nd edge pair and continue the above steps.

The remaining 7 edges consists of $(n - 1)^{th}$ pair in the above n pairs and remaining edges from 1 to 7 substituted in the last ring.

Figure 2: Example for $R(6, 6)$

3 Conclusion

Using the above procedure, it is proved that Roman rings $R(6, n)$ is graceful.

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