Graceful Labeling of Roman Rings
Having Cycle with 6 Vertices

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Abstract

Roman rings can be obtained by introducing n copies of cycle $C_6$ with 6 vertices, which are merged respectively to n teeth of comb graph $P_n \odot L_1$. In this paper, it is proved Roman rings with cycle $C_6$ are graceful.

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1 Introduction

Graphs considered in this paper are simple finite and undirected. In general $G(V, E)$ denotes the graph $G$ with vertex set $V(G)$, edge set $E(G)$, such that $|V(G)| = p$ vertices $|E(G)| = q$ edges. A labeling of the vertices of $G$ with the numbers from 0 to $q$ is an injective map $\phi : V \rightarrow \{0, 1, ... q\}$. A graph $G$ is graceful if there exists a labeling of its vertices such that the map $\phi^* : E \rightarrow \{1, 2, ... q\}$ given by $\phi^*(uv) = |\phi(u) - \phi(v)|$, where $u, v \in V$ and $uv \in E$ is a bijection.

A graph that admits graceful labeling is called graceful graph. The notation
graceful labeling was introduced by Rosa [3] with the name valuation. Gallian [2] gives the extensive survey of contributions to graceful labeling of variety of graphs. Rosa [3], [4] showed that the cycle $C_n$ is graceful if and only if $n = 0$ or $3 \pmod{4}$. Bhat-Nayak and Selvam [1] have shown that the n-cone (also called the n-point suspension of $C_m$) $C_m + K_n$ is graceful when $m = 0$ or $3 \pmod{12}$. They also proved the gracefulness of $C_4 + K_n$, $C_5 + K_2$, $C_7 + K_n$, $C_9 + K_2$, $C_{11} + K_n$ and $C_{19} + K_n$. Seo [5] proved Gracefulness of the union of cycles and paths.

## 2 Main Result

Let $R_1$, $R_2$, ..., $R_n$ be $n$ copies of cycle $C_6$ (we term here as rings). Let the supporting points on the $n$ rings $R_1$, $R_2$, ..., $R_n$ be $t_1$, $t_2$, ..., $t_n$, which are merged respectively to $n$ teeth of comb graph $P_n \odot L_1$. Let $b_1$, $b_2$, $b_3$, ..., $b_n$ be base points of the comb graph from which $n$ rings of equal length say $m$, are hanging, each of which at a tooth of $n$ teeth respectively. The resulting structure is called Roman rings $R(6, n)$. Let the points of $i^{th}$ ring be $c_{i1}^1$, $c_{i2}^1$, ..., $c_{im-1}^1$ for $i = 1, 2, ..., n$.

From the above definition of $R(6, n)$ graph, $|V(R(6, n))| = 7n$. Also, the number of edges of $R(6, n)$ is $|E(R(6, n))| = 8n - 1$.

**Theorem 2.1.** The Roman rings $R(6, n)$ is graceful.

![Figure 1: General form of R (6, n)](image)

The labeling of $(n - 1)$ rings are as follows:-

**Step 1 :** $\phi (b_1) = 0$, $\phi (t_1) = q$. $\phi (c_{11}^1) = 1$, $\phi (c_{21}^1) = 2$.

**Step 2 :** $\phi (b_2) = q - 3$, $\phi (t_2) = 3$. $\phi (c_{12}^2) = q - 1$, $\phi (c_{22}^2) = q - 2$.

(Let $n = 2\lambda + 1$ for odd and $n = 2\lambda$ for even)

**Case 1: n is odd,** then
Step 3 : \( \phi(b_{2i+1}) = \phi(b_{2i-1}) + 4, 1 \leq i \leq \lambda - 1. \)

Step 4 : \( \phi(t_{2i+1}) = q - \phi(b_{2i+1}), 1 \leq i \leq \lambda - 1. \)

Step 5 : \( \phi(b_{2i+2}) = \phi(b_{2i}) - 4, 1 \leq i \leq \lambda - 1. \)

Step 6 : \( \phi(t_{2i+2}) = \phi(t_{2i}) + 4, 1 \leq i \leq \lambda - 1. \)

Case 2: \( n \) is even, then

Step 3 : \( \phi(b_{2i+1}) = \phi(b_{2i-1}) + 4, 1 \leq i \leq \lambda - 1. \)

Step 4 : \( \phi(t_{2i+1}) = q - \phi(b_{2i+1}), 1 \leq i \leq \lambda - 1. \)

Step 5 : \( \phi(b_{2i+2}) = \phi(b_{2i}) - 4, 1 \leq i \leq \lambda - 1. \)

Step 6 : \( \phi(t_{2i+2}) = \phi(t_{2i}) + 4, 1 \leq i \leq \lambda - 1. \)

Step 7 : \( \phi(c_3^{1}) = (n - 2) 2 + 12, \phi(c_3^{1}) = \phi(c_3^{1}) + 2. \)

Step 8 : \( \phi(c_3^{2}) = (n - 3) 6 + 7, \phi(c_3^{2}) = \phi(c_3^{2}) + 2. \)

Case 1: \( n \) is odd (continue), then

Step 9 : \( \phi(c_3^{2i+1}) = \phi(c_3^{2i-1}) + 8, 1 \leq i \leq \lambda - 1. \)

Step 10 : \( \phi(c_4^{2i+1}) = \phi(c_4^{2i-1}) + 8, 1 \leq i \leq \lambda - 1. \)

Step 11 : \( \phi(c_3^{2i+2}) = \phi(c_3^{2i}) - 8, 1 \leq i \leq \lambda - 1. \)

Step 12 : \( \phi(c_4^{2i+2}) = \phi(c_4^{2i}) - 8, 1 \leq i \leq \lambda - 1. \)

Case 2: \( n \) is even (continue), then

Step 9 : \( \phi(c_3^{2i+1}) = \phi(c_3^{2i-1}) + 8, 1 \leq i \leq \lambda - 1. \)

Step 10 : \( \phi(c_4^{2i+1}) = \phi(c_4^{2i-1}) + 8, 1 \leq i \leq \lambda - 1. \)

Step 11 : \( \phi(c_3^{2i+2}) = \phi(c_3^{2i}) - 8, 1 \leq i \leq \lambda - 1. \)

Step 12 : \( \phi(c_4^{2i+2}) = \phi(c_4^{2i}) - 8, 1 \leq i \leq \lambda - 1. \)

The remaining \((n - 1) 2 + 7\) edges starting from 1 to \(2(n - 3) + 12\). Let \( e_i = 2(n - 3) + 12 \). The remaining edges formed in \( n \) pair relations starting from \((e_i, e_i - 2), (e_i - 1, e_i - 3), (e_i - 4, e_i - 6), (e_i - 5, e_i - 7), \ldots \). The labeling of \( \phi(c_5) \) are as follows :-

Step 13 : select maximum edge value pair among \((\phi(c_3^{i}), \phi(c_4^{i}))\), \( i = 1 \) to \( n - 1 \).

Step 14 : substitute maximum edge value pair and the resultant vertex lies in \( 3(n - 2) + 1 \) to \( q - 2(n - 1) \).
Step 15: if not continue with other decreasing order of vertex pairs.

Step 16: select 2nd edge pair and continue the above steps.

The remaining 7 edges consists of \((n-1)\)th pair in the above \(n\) pairs and remaining edges from 1 to 7.

Last ring consists of 6 edges.

Now, induced edge labeling are as follows :-

1: \(\phi^* (b_1t_1) = q\). 2: \(\phi^* (t_1c_1^1) = q - 1\). 3: \(\phi^* (t_1c_2^1) = q - 2\).

4: \(\phi^* (b_1b_2) = q - 3\). 5: \(\phi^* (t_2c_1^2) = q - 4\). 6: \(\phi^* (t_2c_2^2) = q - 5\).

7: \(\phi^* (b_2t_2) = q - 6\). 8: \(\phi^* (b_2b_3) = q - 7\).

Case 1: \(n\) is odd, then

Induced edge labeling of odd segment are as follows :-

9: \(\phi^* (b_{2d+1}t_{2d+1}) = q - 8d, 1 \leq d \leq \lambda - 1\).

10: \(\phi^* (t_{2d+1}c_{1}^{2d+1}) = q - 1 - 8d, 1 \leq d \leq \lambda - 1\).

11: \(\phi^* (t_{2d+1}c_{2}^{2d+1}) = q - 2 - 8d, 1 \leq d \leq \lambda - 1\).

12: \(\phi^* (b_{2d+1}b_{2d+2}) = q - 3 - 8d, 1 \leq d \leq \lambda - 1\).

Induced edge labeling of even segment are as follows :-

13: \(\phi^* (t_{2d+2}c_{1}^{2d+2}) = q - 4 - 8d, 1 \leq d \leq \lambda - 1\).

14: \(\phi^* (t_{2d+2}c_{2}^{2d+2}) = q - 5 - 8d, 1 \leq d \leq \lambda - 1\).

15: \(\phi^* (b_{2d+1}t_{2d+2}) = q - 6 - 8d, 1 \leq d \leq \lambda - 1\).

16: \(\phi^* (b_{2d+2}b_{2d+3}) = q - 7 - 8d, 1 \leq d \leq \lambda - 1\).

Case 2: \(n\) is even, then

Induced edge labeling of odd segment are as follows :-

9: \(\phi^* (b_{2d+1}t_{2d+1}) = q - 8d, 1 \leq d \leq \lambda - 1\).

10: \(\phi^* (t_{2d+1}c_{1}^{2d+1}) = q - 1 - 8d, 1 \leq d \leq \lambda - 1\).

11: \(\phi^* (t_{2d+1}c_{2}^{2d+1}) = q - 2 - 8d, 1 \leq d \leq \lambda - 1\).

12: \(\phi^* (b_{2d+1}b_{2d+2}) = q - 3 - 8d, 1 \leq d \leq \lambda - 2\).

Induced edge labeling of even segment are as follows :-

13: \(\phi^* (t_{2d+2}c_{1}^{2d+2}) = q - 4 - 8d, 1 \leq d \leq \lambda - 2\).

14: \(\phi^* (t_{2d+2}c_{2}^{2d+2}) = q - 5 - 8d, 1 \leq d \leq \lambda - 2\).
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15: \( \phi^*(b_{2d+1}t_{2d+2}) = q - 6 - 8d, 1 \leq d \leq \lambda - 2. \)

16: \( \phi^*(b_{2d+2}b_{2d+3}) = q - 7 - 8d, 1 \leq d \leq \lambda - 2. \)

**Step a**: select maximum edge value from the available assigned values of pair adjacent vertices \( ((c^i_3), (c^i_4)), i = 1 \) to \( n - 1. \)

**Step b**: substitute maximum edge value pair and the resultant vertex value lies in the vertex set from \( 3(n - 2) + 1 \) to \( q - 2(n - 1). \)

**Step c**: if not continue with other decreasing order of vertex pairs.

**Step d**: select 2nd edge pair and continue the above steps.

The remaining 7 edges consists of \( (n - 1)^{th} \) pair in the above \( n \) pairs and remaining edges from 1 to 7 substituted in the last ring.

![Figure 2: Example for R(6, 6)](image)

3 Conclusion

Using the above procedure, it is proved that Roman rings R(6, n) is graceful.

References


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