Center Manifold Analysis of a Delayed-Energy-Based Model

Luca Guerrini

Department of Management
Polytechnic University of Marche, Italy

Abstract

In this paper, the model presented in Bianca et al. [2] has been considered. Based on their analysis of the existence of Hopf bifurcation, by applying the normal form theory and center manifold theory, some explicit formulae for determining the stability and the direction of the Hopf bifurcation periodic solutions are obtained.

Mathematics Subject Classification: 34K18, 91B62

Keywords: Time delay; Dalgaard-Strulik; Hopf bifurcation

1 Introduction

Dalgaard and Strulik [3] developed a model based on the main assumption that there exists a supply relation between the electricity consumption per capita (viewed as the economic counterpart to metabolism) and capital per capita (viewed as the counterpart to body size). From a mathematical point of view, their model of an economy viewed as a transportation network for electricity is isomorphic to the Solow-Swan model [10-11]. Bianca et al. [2] generalized the Dalgaard and Strulik model by considering the energy conservation equation to contain a time delay $T$ (see e.g. [1],[4-8] for a similar approach in other models), which take cares of the previous occurring dynamics. Specifically, if $\xi$ is the energy requirement to operate and maintain the generic capital good $k$, while $\nu$ is the energy costs to create a new capital good, the energy conservation
equation is described by \( e(t) = \xi k(t - T) + dk(t)/dt \). Consequently, the law of motion for capital in Dalgaard and Strulik [3] is described by the following non-linear delay differential equation

\[
\frac{dk(t)}{dt} = \frac{\varepsilon}{\nu} [k(t - T)]^a - \frac{\xi}{\nu} k(t - T).
\]  

(1)

for given initial function \( k(t) = \phi(t), t \in [-T, 0] \), where \( 0 < a < 1 \) is a real constant proportional to the dimension and efficiency of the network, and \( \varepsilon > 0 \) is a real constant, in the sense that it is independent of capital per worker. Equilibria of Eq. (1) coincide with the corresponding points for zero delay, \( T = 0 \). Hence, there exists a unique positive steady state \( k^* \) satisfying the relation \( \varepsilon k^* a - 1 = \xi \). By choosing time delay as a bifurcation parameter, Bianca et al. [2] proved that, as the delay \( T \) increases, there exists a positive number \( T_0 \) such that the equilibrium \( k^* \) is asymptotically stable for \( T_0 > 0 \) and unstable for \( T_0 < 0 \), i.e. the positive equilibrium loses its stability and a Hopf bifurcations occurs. In this paper, based on the analysis of the existence of the Hopf bifurcation, by using the center manifold theory and the normal form method, an explicit algorithm for determining the direction of the Hopf bifurcation and the stability of the bifurcating periodic solution is derived. In this way, one can obtain the important quantities determining the properties of bifurcating periodic solution at the critical value.

2 Direction and stability of Hopf bifurcation

According to Bianca et al. [2], the model (1) undergoes Hopf bifurcation when \( T = T_0 \). As described in Hassard et al. [9], the periodic solution obtained can be further investigated by employing the normal form and center manifold theory. For simplicity, we assume \( \nu = 1 \) in (1). Let \( T = T_0 + \mu, \mu \in \mathbb{R} \), so that Hopf bifurcation occurs at \( \mu = 0 \). Letting \( x = k - k^* \), Eq. (1), expanded in the neighborhood of the null solution up to third order, can be written as

\[
\dot{x} = (\alpha - 1)\xi x(t - T) + \frac{\alpha(\alpha - 1)\xi k^*^{a-1}}{2}x(t - T)^2 \\
+ \frac{\alpha(\alpha - 1)(\alpha - 2)\xi k^*^{a-2}}{6}x(t - T)^3 + \cdots.
\]

(2)

The delayed equation Eq. (2) can be rewritten as a functional differential equation in \( C([-T_0, 0], \mathbb{R}) \), the Banach space of continuous real-valued functions that map \([-T_0, 0]\) into \( \mathbb{R} \). One has

\[
\dot{x}(t) = L_\mu (x_t) + F(\mu, x_t),
\]

(3)

where \( x_t = x(t + \theta) \in C([-T_0, 0], \mathbb{R}) \), and for \( \varphi \in C([-T_0, 0], \mathbb{R}) \) we have

\[
L_\mu \varphi = (\alpha - 1)\xi \varphi(-T)
\]
Center manifold analysis of a delayed-energy-based model

\[ F(\mu, \varphi) = \frac{\alpha(\alpha - 1)\xi k^{-1}}{2} \varphi(-T)^2 + \frac{\alpha(\alpha - 1)(\alpha - 2)\xi k^{-2}}{6} \varphi(-T)^3 + \ldots. \]

Using the Riesz representation theorem, there exists a bounded variation function \( \eta(\theta, \mu) \) for \( \theta \in [-T_0, 0] \) such that

\[ L_{\mu} \varphi = \int_{-T_0}^{0} d\eta(\theta, \mu) \varphi(\theta). \]

For \( \varphi \in C^1([-T_0, 0], \mathbb{R}) \), define

\[ A(\mu) \varphi = \begin{cases} \frac{d\varphi}{d\theta}, & -T_0 \leq \theta < 0, \\ \int_{-T_0}^{0} d\eta(t, \mu) \varphi(t), & \theta = 0, \end{cases} \]

and

\[ R(\mu) \varphi = \begin{cases} 0, & -T_0 \leq \theta < 0, \\ F(\mu, \varphi), & \theta = 0. \end{cases} \]

Then Eq. (3) rewrites as

\[ \dot{x}(t) = A(\mu)x_t + R(\mu)x_t. \quad (4) \]

For \( \psi \in C([0, T_0], \mathbb{R}) \), it is defined

\[ A^*(\mu)\psi(s) = \begin{cases} -\frac{d\psi}{ds}, & 0 < s \leq T_0, \\ \int_{-T_0}^{0} d\eta(t, \mu) \psi(-t), & s = 0, \end{cases} \]

and a bilinear form

\[ <\psi, \varphi> = \tilde{\psi}(0)\varphi(0) - \int_{\theta=-T_0}^{0} \int_{s=0}^{\theta} \tilde{\psi}(s-\theta)d\eta(\theta)\varphi(s)ds, \quad (5) \]

where \( A(0) \) and \( A^* \) are adjoint operators. It is easy to see that \( q(\theta) = e^{i\omega_0 \theta} \), \( -T_0 \leq \theta \leq 0 \), is the eigenvector of \( A(0) \) associated with \( i\omega_0 \), and \( q^*(\theta) = Be^{i\omega_0 \theta} \), \( 0 \leq \theta \leq T_0 \), is the eigenvector of \( A^*(0) \) associated with \(-i\omega_0 \), with

\[ B = \frac{1}{1 + (\alpha - 1)\xi T_0 e^{i\omega_0 T_0}}, \]

and \( <q^*, q> = 1, <q^*, \tilde{q}> = 0 \). Next, we compute the coordinates to describe the center manifold at \( \mu = 0 \). Let \( x_t \) be the solution of (4) when \( \mu = 0 \). Define 

\[ z = <q^*, x_t>, w(t, \theta) = x_t(\theta) - 2Re \{ z q(\theta) \}. \]

We find

\[ \dot{z}(t) = i\omega_0 z(t) + \tilde{q}^*(0) F(0, w(z, \bar{z}) + 2Re \{ z q(0) \}) 
= i\omega_0 z(t) + \tilde{q}^*(0) f_0(z, \bar{z}), \quad (6) \]
with
\[ f_0(z, \bar{z}) = q^*(0) F(0, w(z, \bar{z}) + 2Re\{zq(0)\}) . \] \quad (7)

We rewrite Eq. (6) as
\[ \dot{z}(t) = i\omega_0 z(t) + g(z, \bar{z}), \] \quad (8)

where
\[ w(z, \bar{z}) = w_{20} \frac{z^2}{2} + w_{11} z \bar{z} + w_{02} \frac{\bar{z}^2}{2} + \cdots, \] \quad (9)

and
\[ g(z, \bar{z}) = g_{20} \frac{z^2}{2} + g_{11} z \bar{z} + g_{02} \frac{\bar{z}^2}{2} + g_{21} z^2 \bar{z} + \cdots. \] \quad (10)

It follows from (4) and (8) that
\[
\dot{w} = \begin{cases} 
Aw - 2Re\{q^*(0)fo\theta\}, & -T_0 \leq \theta < 0, \\
Aw - 2Re\{q^*(0)fo\theta\} + f_0, & \theta = 0,
\end{cases} \quad (11)
\]

where
\[ H(z, \bar{z}) = H_{20} \frac{z^2}{2} + H_{11} z \bar{z} + H_{02} \frac{\bar{z}^2}{2} + \cdots. \] \quad (12)

Since \( x_t(\theta) = w(z, \bar{z})(\theta) + z e^{i\omega_0 \theta} + \bar{z} e^{-i\omega_0 \theta} \), we get \( x_t(-T_0) = w(z, \bar{z})(-T_0) + z e^{-i\omega_0 T_0} + \bar{z} e^{i\omega_0 T_0} \). Then, we derive that the function \( f_0 \) in (7) can be expressed as
\[
f_0 = \frac{\alpha(\alpha - 1) \xi k_{x}^{-1}}{2} \left[ w(z, \bar{z})(-T_0) + z e^{-i\omega_0 T_0} + \bar{z} e^{i\omega_0 T_0} \right]^2
+ \frac{\alpha(\alpha - 1)(\alpha - 2) \xi k_{x}^{-2}}{6} \left[ w(z, \bar{z})(-T_0) + z e^{-i\omega_0 T_0} + \bar{z} e^{i\omega_0 T_0} \right]^3 + \cdots
\]
\[
= \frac{\alpha(\alpha - 1) \xi k_{x}^{-1}}{2} \left[ z^2 e^{-2i\omega_0 T_0} + \bar{z}^2 e^{2i\omega_0 T_0} + 2z \bar{z}
+ 2w(z, \bar{z})(-T_0)ze^{-i\omega_0 T_0} + 2\bar{z} e^{i\omega_0 T_0} w(z, \bar{z})(-T_0) + \cdots \right]
+ \frac{\alpha(\alpha - 1)(\alpha - 2) \xi k_{x}^{-2}}{6} \left[ 3z^2 \bar{z} e^{-i\omega_0 T_0} + 3z \bar{z}^2 e^{i\omega_0 T_0} + \cdots \right] + \cdots
\]
\[
= \frac{\alpha(\alpha - 1) \xi k_{x}^{-1}}{2} \left[ z^2 e^{-2i\omega_0 T_0} + \bar{z}^2 e^{2i\omega_0 T_0} + 2z \bar{z} + 2e^{-i\omega_0 T_0} w_{11}(-T_0)z^2 \bar{z}
+ e^{-i\omega_0 T_0} w_{02}(-T_0)z \bar{z}^2 + e^{i\omega_0 T_0} w_{20}(-T_0)z^2 \bar{z} + 2e^{i\omega_0 T_0} w_{11}(-T_0)z \bar{z}^2 + \cdots \right]
+ \frac{\alpha(\alpha - 1)(\alpha - 2) \xi k_{x}^{-2}}{6} \left[ 3z^2 \bar{z} e^{-i\omega_0 T_0} + 3z \bar{z}^2 e^{i\omega_0 T_0} + \cdots \right] + \cdots
\]
Comparing coefficients with (12), the following hold

\[
g(z, \bar{z}) = \bar{q}^*(0) f_0 = \bar{B} f_0 = \bar{B} \frac{\alpha - 1}{2} \xi k^{-1} e^{-2i\omega T_0} z^2
\]

Thus

\[
g(z, \bar{z}) = \bar{q}^*(0) f_0 = \bar{B} f_0 = \bar{B} \frac{\alpha - 1}{2} \xi k^{-1} e^{-2i\omega T_0} z^2
\]

Comparing coefficients with (12), the following hold

\[
g_{20} = B \alpha - 1 \xi k^{-1} e^{-2i\omega T_0} = -B \alpha - 1 \xi k^{-1}
\]

\[
g_{11} = \bar{B} \alpha - 1 \xi k^{-1}
\]

\[
g_{02} = \bar{B} \alpha - 1 \xi k^{-1} e^{2i\omega T_0} = -\bar{B} \alpha - 1 \xi k^{-1}
\]

\[
g_{21} = 2i \bar{B} \left[ -\alpha - 1 \xi k^{-1} w_{11}(-T_0) + \frac{\alpha - 1}{2} \xi k^{-1} e^{-2i\omega T_0} w_{20}(-T_0) \right].
\]

For \(-T_0 \leq \theta < 0,\)

\[
H(z, \bar{z}) = -2Re \{ \bar{q}^*(0) f_0 q(\theta) \} = -gq(\theta) - \bar{g} q(\theta)
\]

\[
= - \left( \frac{g_{20} z^2}{2} + g_{11} z \bar{z} + \frac{g_{02} z^2}{2} + \cdots \right) q(\theta)
\]

\[
- \left( \frac{\bar{g}_{20} z^2}{2} + \bar{g}_{11} z \bar{z} + \frac{\bar{g}_{02} z^2}{2} + \cdots \right) \bar{q}(\theta).
\]
Hence, (10) yields
\[ H_{20} = -g_{20}q(\theta) - \bar{g}_{02}q(\theta), \quad H_{11} = -g_{11}q(\theta) - \bar{g}_{11}q(\theta). \] (13)

From \( \dot{w}(z, \bar{z}) = w_z \dot{z} + w_{\bar{z}} \dot{\bar{z}} \), recalling (8) and (9), we can get a second expression for \( \dot{w} \), which compared with (11) leads to the following equations for the coefficients \( w_{ij} \):
\[ (2i\omega_0 - A) w_{20}(\theta) = H_{20}(\theta), \quad Aw_{11}(\theta) = -H_{11}(\theta). \] (14)

From (13) and (14), we have
\[ \dot{w}_{20} = 2i\omega_0 w_{20} + g_{20}e^{i\omega_0 \theta} + \bar{g}_{02}e^{-i\omega_0 \theta}. \]

Solving for \( w_{20} \), and similarly for \( w_{11} \), we obtain
\[ w_{20}(\theta) = -\frac{g_{20}}{i\omega_0} e^{i\omega_0 \theta} - \frac{\bar{g}_{02}}{3i\omega_0} e^{-i\omega_0 \theta} + E_1 e^{2i\omega_0 \theta}, \] (15)
and
\[ w_{11}(\theta) = \frac{g_{11}}{i\omega_0} e^{i\omega_0 \theta} - \frac{\bar{g}_{11}}{i\omega_0} e^{-i\omega_0 \theta} + E_2, \]
respectively. Here, \( E_1 \) and \( E_2 \) are constants to be determined by setting \( \theta = 0 \) in \( H \). A direct computation shows
\[ E_1 = \frac{3g_{20} - \bar{g}_{02}}{3\omega_0} - \frac{g_{20} + \bar{g}_{02} + \alpha(\alpha - 1)|\xi|}{(\alpha - 1)\xi - 2i\omega_0} k_*^{-1}, \]
\[ E_2 = \frac{g_{11} + \bar{g}_{11} - \alpha(\alpha - 1)\mu k_*^{-1}}{(\alpha - 1)\xi} + \frac{g_{11} + \bar{g}_{11}}{\omega_0}. \]

In conclusion, all \( g_{ij} \) have been obtained, and thus we can compute the quantities
\[ C_1(0) = \frac{i}{2\omega_0} \left[ g_{20}g_{11} - 2 |g_{11}|^2 - \frac{|g_{02}|^2}{3} \right] + \frac{g_{21}}{2}, \]
\[ \mu_2 = -\frac{ReC_1(0)}{Re\lambda'(0)}, \quad T_2 = -\frac{ImC_1(0) + \mu_2 Im\lambda'(0)}{\omega_0}, \quad \beta_2 = 2ReC_1(0), \]
which determine the properties of bifurcating periodic solutions at the critical value \( T_0 \). From the discussion above, we have the following result.

**Theorem 2.1.** Let \( k_* \) be the unique positive solution of the model (1).

1) \( \mu_2 \) determines the direction of the Hopf bifurcation when \( T > T_0 \): if \( \mu_2 > 0 \) (resp. \( \mu_2 < 0 \)), then the Hopf bifurcation is supercritical (resp. subcritical) and the bifurcating periodic solution exists for \( T > T_0 \) (resp. \( T < T_0 \)) in a sufficiently small \( T_0 \)-neighbourhood.
2) \( \beta_2 \) determines the stability of the bifurcating periodic solution: if \( \beta_2 < 0 \) (resp. \( \beta_2 > 0 \)), the bifurcating periodic solution is locally asymptotically stable (resp. unstable).

3) \( T_2 \) determines the period of the bifurcating periodic solution: if \( T_2 > 0 \) (resp. \( T_2 < 0 \)), the period increases (resp. decreases).

References


Received: April 29, 2015; Published: July 2, 2015