On Algorithm of Finding Solutions of Semiperiodical Boundary Value Problem for Linear Hyperbolic Equation and its Convergence

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Abstract

The algorithm of finding approximate solutions of semiperiodical boundary value problem for linear hyperbolic equation is constructed. Sufficient conditions of existence of solutions and convergence of constructed method of finding approximate solutions of semiperodical boundary value problem for linear hyperbolic equation with two variables are obtained.

Keywords: linear hyperbolic equation; semiperiodical boundary value problem; modification of Euler method; approximate solution

1 Introduction

We consider the following semiperiodical boundary value problem for linear hyperbolic equation of two variables

\[
\frac{\partial^2 u}{\partial x \partial t} = A(x, t) \frac{\partial u}{\partial x} + B(x, t) \frac{\partial u}{\partial t} + C(x, t) u + f(x, t),
\]

\[u(0, t) = \psi(t), \quad t \in [0, T],\]

\[u(x, 0) = u(x, T), \quad x \in [0, \omega],\]
in the bounded set $\Omega = [0, \omega] \times [0, T]$. Here $A(x, t)$, $B(x, t)$, $C(x, t)$, $f(x, t)$ are continuous functions from $\Omega$ to $R$, and $\psi(t)$ is a continuously differentiable function of $[0, T]$ to $R$, such that $\psi(0) = \psi(T)$. One of the basic and well-studied problems of the theory of the second order hyperbolic equations is the periodic boundary problem. The study of periodical boundary value problem for hyperbolic equation with mixed derivatives had begun with the work of L. Cesari [10-12]. Future it was developed in the work of J.K. Hale [16], G. Hesquet [17], A.K. Aziz [6], A.K. Aziz A.M. Meyers [9], A.K. Aziz M.G. Horak [8], A.K. Aziz and S.L. Brodsky [7], V. Lakshmikantham S.G. Pandit [28], S.V. Zhestkov [38], A.M. Samoylenko [36], T.I. Kiguradze. The conditions of existence of the periodical solutions of higher order hyperbolic equations are studied in the work of B.I. Ptashnik [35]. Y.A. Mitropolsky, G.P. Homa, M.I. Gromiyak [31] used asymptotical methods to study periodical solutions of hyperbolic equations. Fourier method, method of successive approximations, functional method and the variation method were used for finding periodic solutions of boundary value problems for second order hyperbolic equations [15], [29], [30], [36], [37]. The boundary value problem with nonlocal conditions for hyperbolic equations were studied by L.B. Nakhushev [33], [34], Ju. A. Mitropolsky and L.B. Urmancheva [32], S.S. Kharibegashvili [26], [27], T.I. Kiguradze [21]-[25], L.S. Pulkina [14] and others. Later the theory of nonlocal boundary value problems for system of hyperbolic equations with mixed derivatives was developed in the works of D.S. Dzhumabaev, A.T. Asanova and their students. By A.T. Asanova [1], [2] on the basis of parametrization method, proposed by D.S. Dzhumabaev [13] to study boundary value problems for system of ordinary differential equation, the introducing functional parameters method was developed for study boundary value problems with data on characteristics for hyperbolic system with mixed derivatives. The algorithm of finding classical solutions of boundary value problem with data on characteristics for hyperbolic system with mixed derivatives was constructed and coefficient features of unique solvability of given problem was obtained. The criterion for correctly solvability given nonlocal boundary value problem [3]-[5] was established. Herewith one of the basic item of the algorithm is finding the solutions of the family nonlocal boundary value problems for ordinary differential equations.

The construction of approximate solutions of these problems, when the parameter of family changes continuously, leads to great and sometimes to insurmountable difficulties.

Therefore there is a need to create effective algorithms for finding the solutions of semiperiodical boundary value problems for nonlinear hyperbolic equations, that do not use the family of boundary value problems for ordinary differential equations, where the parameter of family, that is variable, changes continuously.

One of the well-known and effective numerical method of solving Cauchy
problem for nonlinear ordinary differential equations is the Euler method. Beside the existence of the solution of the problem, the Euler method gives the direct algorithm of constructing the approximate solution, arbitrarily close to the exact solution.

The sufficient conditions of existence of "isolated" solution is obtained in [18], and the modification Euler method for finding initial approximation of semiperiodical boundary value problem for nonlinear hyperbolic equation with mixed derivatives is proposed in the works [19], [20].

In this article we study the boundary value problem (1)-(3) by applying the modified Euler method. In Euler method integral curve, which is the solution of nonlinear ordinary differential equation with the initial condition has been approximatively replaced successively step by step some line segments. In this case differential equation considered interval is devided into n partition. In each step the initial problem for differential equation with fixed right-hand side is solved. By modified Euler method the rectangle region, where given the linear hyperbolic equation is divided into subsets with some step size $h > 0$ by variable $x$ and at each step the periodical boundary value problem for ordinary differential equations are solved.

We introduce the following notations and definitions. We denote by $C(\Omega)$ the class of continuous real valued functions $u : \Omega \rightarrow R$ in $\Omega$. We introduce the following norm $\|u(x, \cdot)\|_1 = \max_{t \in [0,T]} |u(x, t)|$ for each fixed value of $x \in [0, \omega]$ in $C(\Omega)$. And we denote by $C^1([0,T])$ the space of continuous real valued functions $\psi(t)$ with norm $\|\psi\|_0 = \max (\max_{t \in [0,T]} |\psi(t)|, \max_{t \in [0,T]} |\dot{\psi}(t)|).

We denote by $C_{x,t}^{1,1}(\Omega)$ the space of continuous and continuously differentiable functions $u(x,t)$ in $\Omega$, such that the following norm is finite

$$\|u\|_0 = \max(\max_{\Omega} |u(x,t)|, \max_{\Omega} |u_x(x,t)|, \max_{\Omega} |u_t(x,t)|).$$

**Definition 1.1** A solution of the problem (1)-(3) in the domain $C(\Omega)$ is a function $u(x,t) \in C(\Omega)$, such that the partial derivatives $\frac{\partial u(x,t)}{\partial x} \in C(\Omega)$, $\frac{\partial u(x,t)}{\partial t} \in C(\Omega)$, $\frac{\partial^2 u(x,t)}{\partial x \partial t} \in C(\Omega)$ exist and satisfy the equation (1) for every $(x,t) \in \Omega$, and has the value $\psi(t)$ for all $t \in [0,T]$ on the characteristics $x = 0$, and the values on the characteristics $t = 0$ and $t = T$ are equal for all $x \in [0,\omega]$.

## 2 Transition to the equivalent problem

Now we consider semiperiodical boundary value problem (1)-(3).
Definition 2.1 The boundary value problem (1)-(3) is called correctly solvable, if it has a unique solution \( u(x, t) \) for all \( f(x, t) \in C(\bar{\Omega}) \) and there exists continuously differentiable on \([0, T]\) function \( \psi(t) \) such that

\[
\|u\|_0 \leq K \max \{\|\psi\|_0, \max_{\bar{\Omega}} |f(x, t)|\},
\]

where \( K \) is constant that does not depend on \( f(x, t) \) and \( \psi(t) \).

We introduce the following functions \( v(x, t) = \frac{\partial u(x, t)}{\partial x}, w(x, t) = \frac{\partial u(x, t)}{\partial t} \). Then we reduce problem (1)-(3) to the equivalent problem

\[
\frac{\partial v}{\partial t} = A(x, t)v + B(x, t)w + C(x, t)u + f(x, t), \quad (x, t) \in \bar{\Omega} \tag{1.1}
\]

\[
v(x, 0) = v(x, T), \quad x \in [0, \omega], \tag{1.2}
\]

\[
u(x, t) = \psi(t) + \int_0^x v(\xi, t) d\xi, \quad w(x, t) = \dot{\psi}(t) + \int_0^x v(\xi, t) d\xi \tag{1.3}
\]

Definition 2.2 The triple of continuous functions \( \{u(x, t), w(x, t), v(x, t)\} \) in \( \bar{\Omega} \) is said to be a solution of problem (1.1)-(1.3), if the function \( v(x, t) \in C(\bar{\Omega}) \) is continuously differentiable in \( \bar{\Omega} \) with respect to \( t \) and satisfies the family of periodical boundary value problem (1.1), (1.2), where functions \( u(x, t) \) and \( w(x, t) \) are connected with \( v(x, t) \), \( \frac{\partial v(x, t)}{\partial t} \) by functional relations (1.3).

The problem (1)-(3) and (1.1)-(1.3) are equivalent in the sense if the function \( u(x, t) \) is the solution of problem (1)-(3), then the triple of functions \( \{u(x, t), w(x, t), v(x, t)\} \) is the solution of problem (1.1)-(1.3), and vice versa.

Let function \( u(x, t) \) be a solution of problem (1)-(3). We make up the triple of continuous functions \( \{u(x, t), w(x, t), v(x, t)\} \in \bar{\Omega} \), where \( v(x, t) = \frac{\partial u(x, t)}{\partial x}, w(x, t) = \frac{\partial u(x, t)}{\partial t} \). Since \( u(x, t) = u(0, t) + \int_0^x \frac{\partial u(\xi, t)}{\partial \xi} d\xi = \psi(t) + \int_0^x v(\xi, t) d\xi \) and taking into account that \( u(x, t) \) is the solution of problem (1)-(3), we have

\[
\frac{\partial^2 u(x, t)}{\partial x \partial t} = \frac{\partial^2 u(x, t)}{\partial t \partial x}, \quad w(x, t) = \frac{\partial u(x, t)}{\partial t} = \frac{\partial u(0, t)}{\partial t} + \int_0^x \frac{\partial^2 u(\xi, t)}{\partial \xi \partial t} d\xi = \frac{\partial u(0, t)}{\partial t} + \int_0^x \frac{\partial^2 u(\xi, t)}{\partial t \partial \xi} d\xi = \dot{\psi}(t) + \int_0^x \frac{\partial v(\xi, t)}{\partial \xi} d\xi,
\]

\[
\frac{\partial v}{\partial t} = A(x, t) \frac{\partial u}{\partial x} + B(x, t) \frac{\partial u}{\partial t} + C(x, t) u + f(x, t),
\]

\[
\frac{\partial u(x, 0)}{\partial x} - \frac{\partial u(x, T)}{\partial x} = v(x, 0) - v(x, T) = 0,
\]
i.e. compiled by this way the triple is the solution of problem (1.1)-(1.3). And vice versa, if the triple of functions \( \{u(x,t), w(x,t), v(x,t)\} \) is the solution of problem (1.1)-(1.3), then functional relations (1.3) imply that the function \( u(x,t) \) satisfies condition \( u(0,t) = \psi(t) \) and has continuously first order partial derivatives
\[
\frac{\partial u(x,t)}{\partial x} = v(x,t), \quad \frac{\partial u(x,t)}{\partial t} = \dot{\psi}(t) + \int_0^x \frac{\partial v(\xi,t)}{\partial t} d\xi = w(x,t), \tag{1.4}
\]
and continuously second order mixed derivatives
\[
\frac{\partial^2 u(x,t)}{\partial t \partial x} = \frac{\partial v(x,t)}{\partial t}, \quad \frac{\partial^2 u(x,t)}{\partial x \partial t} = \frac{\partial v(x,t)}{\partial t}. \tag{1.5}
\]
By substituting (1.4) and (1.5) into (1.1) and (1.2), we obtain that the function \( u(x,t) \) satisfies the hyperbolic equation (1.1) and boundary condition (1.2) for all \((x,t) \in \bar{\Omega}\). Since it satisfies also the initial condition (1.2), then \( u(x,t) \) is the solution of problem (1.1)-(1.3).

We consider the family of periodical boundary problems for ordinary differential equations
\[
\frac{dv}{dt} = A(x,t)v + F(x,t), \quad (x,t) \in \bar{\Omega} \tag{1.6}
\]
\[
v(x,0) = v(x,T), \quad x \in [0,\omega], \tag{1.7}
\]
where \( A(x,t) \) and \( F(x,t) \) are continuous functions in \( \bar{\Omega} \).

**Definition 2.3** Continuously differentiable with respect to \( t \) function \( v(x,t) \) in \( C(\bar{\Omega}) \) is said to be a solution of boundary value problem (1.6), (1.7), if it satisfies the system of equation (1.6) for all \((x,t) \in \bar{\Omega}\) and boundary condition (1.7) for all \( x \in [0,\omega] \).

**Definition 2.4** Boundary value problem (1.4), (1.5) is said to be correctly solvable, if it has a unique solution \( v(x,t) \) for all \( F(x,t) \) and the following inequality holds
\[
\|v(x,\cdot)\|_1 \leq K \|F(x,\cdot)\|_1,
\]
where \( K \) is constant that does not depend on \( F(x,t) \).

Theorem 3 [61, P.23] implies that semiperiodical boundary value problem (1.1)-(1.3) is correctly solvable if and only if periodical boundary value problem (1.6), (1.7) is correctly solvable.

Necessary and sufficient conditions of correctly solvability of problem (1.6), (1.7) were established in the work [10. . 2010.].
3 Modification of Euler method applied to the linear semiperiodical boundary value problem

We use modification of Euler method to find a solution of problem (1.1)-(1.3). We divide the interval $[0, \omega]$ into $N$ subintervals with step size $h > 0$ and $N \omega = \omega$. At each step we solve periodical boundary value problem for system ordinary differential equation.

Step 0. We define the functions $v^{(0)}(t), \dot{v}^{(0)}(t), u^{(0)}(t), w^{(0)}(t)$ by the following way

\[ v^{(0)}(t) = 0, \; \dot{v}^{(0)}(t) = 0, \; u^{(0)}(t) = \psi(t), \; w^{(0)}(t) = \dot{\psi}(t), \; t \in [0, T]. \]

Step 1. By solving periodical boundary value problem

\[
\frac{dv^{(1)}}{dt} = A(0, t)v^{(1)} + B(0, t)w^{(0)}(t) + C(0, t)u^{(0)}(t) + f(0, t), \\
v^{(1)}(0) = v^{(1)}(T),
\]

we find the function $v^{(1)}(t)$.

Step i. Assuming $v^{(i-1)}(t), \dot{v}^{(i-1)}(t), i = 1, N + 1$, are known functions, we find the function $v^{(i)}(t)$ by solving periodical boundary value problem

\[
\frac{dv^{(i)}}{dt} = A((i-1)h, t)v^{(i)} + B((i-1)h, t)\left(\dot{\psi}(t) + h \sum_{j=0}^{i-1} v^{(j)}(t)\right) + \\
+ C((i-1)h, t)\left(\psi(t) + h \sum_{j=0}^{i-1} v^{(j)}(t)\right) + f((i-1)h, t),
\]

\[
v^{(i)}(0) = v^{(i)}(T), \; i = 1, N + 1.
\]

\[
\alpha_1 = \max_{(x,t) \in \Omega} |B(x, t)|, \; \alpha_2 = \max_{(x,t) \in \Omega} |C(x, t)|, \; \hat{f}(t) = \max_{x \in [0, \omega]} |f(x, t)|.
\]

Theorem 3.1 Let inequality $|\int_0^T A(x, \tau)d\tau| \geq \delta > 0$ holds for all $x \in [0, \omega]$. Then periodical boundary value problem for system of linear ordinary differential equations (2.3), (2.4) has a unique solution $\{v^{(i)}(t)\}, i = 1, N + 1$ for all $h > 0 : Nh = \omega$ and there exists a constant $K$ such that the following inequality holds

\[
\max \left(\|v^{(i)}(\cdot)\|_1, \|\dot{v}^{(i)}(\cdot)\|_1\right) \leq K(\alpha, \delta, T)[1 + K(\alpha, \delta, T)(\alpha_1 + \alpha_2) \cdot h]^{\frac{n}{\delta}} \times \\
\times \left((\alpha_1 + \alpha_2) \max \left(\|\psi(\cdot)\|_1, \|\dot{\psi}(\cdot)\|_1 + \|\hat{f}(\cdot)\|_1\right), \; i = 1, N + 1,
\]

where $K(\alpha, \delta, T) = \max \{K_1(\alpha, \delta, T), 1 + \alpha \cdot K_1(\alpha, \delta, T)\}$. 
Proof. By assumptions of the theorem the problem (2.3), (2.4) has a unique solution \{v^{(i)}(t)\}, \(i = 1, N + 1\), that can be represented in the following form

\[ v^{(i)}(t) = \frac{\exp \left( \int_0^t A((i-1)h, \tau)d\tau \right)}{1 - \exp \left( \int_0^T A((i-1)h, \tau)d\tau \right)} \cdot \int_0^T F((i-1)h, \tau) \times \]

\[ \times \exp \left( \int_\tau^T A((i-1)h, \tau)d\tau \right) d\tau + \int_0^T F((i-1)h, \tau) \times \]

\[ \times \exp \left( \int_\tau^T A((i-1)h, \tau)d\tau \right) d\tau, \quad t \in [0, T], \]

where \( F((i-1)h, t) = B((i-1)h, t)(\dot{\psi}(t) + h \sum_{j=0}^{i-1} \dot{v}^{(j)}(t)) + \]

\[ + C((i-1)h, t)(\psi(t) + h \sum_{j=0}^{i-1} v^{(j)}(t)) + f((i-1)h, t). \]

This solution satisfies the following inequality

\[
\|v^{(i)}(\cdot)\|_1 \leq \frac{e^{\alpha T} - 1}{\alpha} \left( \frac{e^\delta}{e^\delta - 1} + 1 \right) \left\{ \left( \|B((i-1)h, \cdot)\|_1 + \|C((i-1)h, \cdot)\|_1 \right) \times \right. \\
\times \left[ h \cdot \sum_{j=0}^{i-1} \max (\|v^{(j)}(\cdot)\|_1, \|\dot{v}^{(j)}(\cdot)\|_1) + \max (\|\psi(\cdot)\|_1, \|\dot{\psi}(\cdot)\|_1) \right] + \right. \\
+ \left\| f((i-1)h, \cdot) \right\|_1 \right\} \leq K_1(\alpha, \delta, T) \left\{ (\alpha_1 + \alpha_2) \cdot h \cdot \sum_{j=0}^{i-1} \max (\|v^{(j)}(\cdot)\|_1, \|\dot{v}^{(j)}(\cdot)\|_1) + \right. \\
+ (\alpha_1 + \alpha_2) \max (\|\psi(\cdot)\|_1, \|\dot{\psi}(\cdot)\|_1) + \| f((i-1)h, \cdot) \right\|_1 \}, \quad i = 1, N + 1. \]

Now we will prove the uniqueness of the solution of the boundary value problem (2.3), (2.4). We suppose that there exist two solutions \( \tilde{v}^{(i)}(t) \) and \( \tilde{\sigma}^{(i)}(t) \) of the problem (2.3), (2.4). We denote by \( \Delta v^{(i)}(t), \Delta v^{(i)}(t) = \tilde{v}^{(i)}(t) - \tilde{\sigma}^{(i)}(t) \) their difference, i.e. \( \Delta v^{(i)}(t) \) the following boundary value problem

\[
\frac{d\Delta v^{(i)}(t)}{dt} = A((i-1)h, t) \Delta v^{(i)} + B((i-1)h, t) \cdot h \cdot \sum_{j=0}^{i-1} \Delta \dot{v}^{(j)} + \]

\[ + C((i-1)h, t) \cdot h \cdot \sum_{j=0}^{i-1} \Delta v^{(j)}, \quad \Delta v^{(i)}(0) = \Delta v^{(i)}(T). \]

Solving above problem, we obtain

\[
\Delta v^{(i)}(t) = \frac{\exp \left( \int_0^t A((i-1)h, \tau)d\tau \right)}{1 - \exp \left( \int_0^T A((i-1)h, \tau)d\tau \right)} \cdot \int_0^T \left[ B((i-1)h, \tau) \cdot h \times \right. \\
\times \left. \exp \left( \int_\tau^T A((i-1)h, \tau)d\tau \right) d\tau \right] \Delta v^{(i)}(T) + \Delta v^{(i)}(0). \]
Then the following inequality holds

\[ \|v^{(i)}(\cdot)\|_1 \leq K_1(\alpha, \delta, T)(\alpha_1 + \alpha_2) \cdot h \cdot \sum_{j=0}^{i-1} \|\Delta v^{(j)}(\cdot)\|_1. \]  

(2.5)

Since by assumption \( v^{(0)}(t) = 0 \), then (2.5) implies

\[ \|\Delta v^{(1)}(\cdot)\|_1 = 0, \text{ i.e. } \bar{v}^{(1)}(t) = \bar{v}^{(1)}(t) \text{ for } i = 1. \]

For function \( \Delta v^{(i)}(t), \quad i = 2, N + 1 \), the following inequality holds

\[ \|\Delta v^{(i)}(\cdot)\|_1 \leq K_1(\alpha, \delta, T)(\alpha_1 + \alpha_2) \cdot h \times \]

\[ \times \left( 1 + K_1(\alpha, \delta, T)(\alpha_1 + \alpha_2) \cdot h \right)^{-i-2} \|\Delta v^{(1)}(\cdot)\|_1 = 0. \]

Hence it follows that \( \Delta v^{(i)}(t) \equiv 0, i = 2, N + 1, \ t \in [0, T] \). Therefore \( \bar{v}^{(i)}(t) = v^{(i)}(t) \) for all \( t \in [0, T], \ i = 1, N + 1 \). Thus, the problem (2.3), (2.4) has a unique solution.

Since, \( v^{(i)}(t) \) satisfies the differential equation (2.3), then for its derivatives the following inequality holds

\[ \|v^{(i)}(\cdot)\|_1 \leq \alpha \|v^{(i)}(\cdot)\|_1 + (\alpha_1 + \alpha_2) \cdot \left[ h \cdot \sum_{j=1}^{i-1} \max \left( \|v^{(j)}(\cdot)\|_1, \|v^{(j)}(\cdot)\|_1 \right) + \right. \]

\[ + \max \left( \|\psi(\cdot)\|_1, \|\dot{\psi}(\cdot)\|_1 \right) + \|f((i-1)h, \cdot)\|_1 = \alpha \|v^{(i)}(\cdot)\|_1 + (\alpha_1 + \alpha_2) \times \]

\[ \left. \times h \cdot \sum_{j=0}^{i-1} \max \left( \|v^{(j)}(\cdot)\|_1, \|v^{(j)}(\cdot)\|_1 \right) + (\alpha_1 + \alpha_2) \max \left( \|\psi(\cdot)\|_1, \|\dot{\psi}(\cdot)\|_1 \right) + \right. \]

\[ + \left. \|f((i-1)h, \cdot)\|_1. \]  

(2.6)

Substituting \( \|v^{(i)}(\cdot)\|_1 \) into the right-hand side of (2.6), we obtain

\[ \|v^{(i)}(\cdot)\|_1 \leq (1 + \alpha \cdot K_1(\alpha, \delta, T)) \left[ (\alpha_1 + \alpha_2) \max \left( \|\psi(\cdot)\|_1, \|\dot{\psi}(\cdot)\|_1 \right) + \right. \]

\[ + (\alpha_1 + \alpha_2) \cdot h \cdot \sum_{j=0}^{i-1} \max \left( \|v^{(j)}(\cdot)\|_1, \|v^{(j)}(\cdot)\|_1 \right) + \|f((i-1)h, \cdot)\|_1. \]
Thus, we have
\[
\max (\|v^{(i)}(\cdot)\|_1, \|\dot{v}^{(i)}(\cdot)\|_1) \leq \\
\leq K(\alpha, \delta, T) \cdot \left[ (\alpha_1 + \alpha_2) \cdot h \cdot \sum_{j=0}^{i-1} \max (\|v^{(j)}(\cdot)\|_1, \|\dot{v}^{(j)}(\cdot)\|_1) + \\
+ (\alpha_1 + \alpha_2) \max (\|\psi(\cdot)\|_1, \|\dot{\psi}(\cdot)\|_1) + \|f((i-1)h, \cdot)\|_1 \right], \quad i = 1, N + 1. \tag{2.7}
\]

For \(i = 1\) inequality (2.7) implies
\[
\max (\|v^{(i)}(\cdot)\|_1, \|\dot{v}^{(i)}(\cdot)\|_1) \leq \\
\leq K(\alpha, \delta, T) \left( (\alpha_1 + \alpha_2) \max (\|\psi(\cdot)\|_1, \|\dot{\psi}(\cdot)\|_1) + \|\hat{f}(\cdot)\|_1 \right).
\]

By virtue of estimates of functions \(v^{(i)}(t), \dot{v}^{(i)}(t)\) for \(i = 2\), inequality (2.7) implies that
\[
\max (\|v^{(2)}(\cdot)\|_1, \|\dot{v}^{(2)}(\cdot)\|_1) \leq K(\alpha, \delta, T) \left( (\alpha_1 + \alpha_2) \cdot h \cdot \max (\|v^{(1)}(\cdot)\|_1, \|\dot{v}^{(1)}(\cdot)\|_1) + \\
+ (\alpha_1 + \alpha_2) \max (\|\psi(\cdot)\|_1, \|\dot{\psi}(\cdot)\|_1) + \|\hat{f}(\cdot)\|_1 \right) \leq K(\alpha, \delta, T) \left( (\alpha_1 + \alpha_2) \cdot h \times \\
\times K(\alpha, \delta, T) \left( (\alpha_1 + \alpha_2) \max (\|\psi(\cdot)\|_1, \|\dot{\psi}(\cdot)\|_1) + \|\hat{\dot{f}}(\cdot)\|_1 \right) + \\
+ (\alpha_1 + \alpha_2) \max (\|\psi(\cdot)\|_1, \|\dot{\psi}(\cdot)\|_1) + \|\hat{f}(\cdot)\|_1 \right) \leq [1 + K(\alpha, \delta, T)(\alpha_1 + \alpha_2) \cdot h \times \\
\times K(\alpha, \delta, T) \left( (\alpha_1 + \alpha_2) \max (\|\psi(\cdot)\|_1, \|\dot{\psi}(\cdot)\|_1) + \|\hat{f}(\cdot)\|_1 \right).\]

By arguing as in (2.7) for \(v^{(3)}(t), \dot{v}^{(3)}(t)\), we obtain
\[
\max (\|v^{(3)}(\cdot)\|_1, \|\dot{v}^{(3)}(\cdot)\|_1) \leq [1 + K(\alpha, \delta, T)(\alpha_1 + \alpha_2) \cdot h]^2 \cdot K(\alpha, \delta, T) \times \\
\times \left( (\alpha_1 + \alpha_2) \max (\|\psi(\cdot)\|_1, \|\dot{\psi}(\cdot)\|_1) + \|\hat{f}(\cdot)\|_1 \right).
\]

Repeating this process for all \(i = 1, N + 1\), we establish that
\[
\max (\|v^{(i)}(\cdot)\|_1, \|\dot{v}^{(i)}(\cdot)\|_1) \leq \\
\leq [1 + K(\alpha, \delta, T)(\alpha_1 + \alpha_2) \cdot h]^{i-1} \cdot K(\alpha, \delta, T) \left( \|\hat{\dot{f}}(\cdot)\|_1 + (\alpha_1 + \alpha_2) \times \\
\times \max (\|\psi(\cdot)\|_1, \|\dot{\psi}(\cdot)\|_1) \right) \leq K(\alpha, \delta, T)[1 + K(\alpha, \delta, T)(\alpha_1 + \alpha_2) \cdot h]^{\frac{N}{2}} \times \\
\times \left( (\alpha_1 + \alpha_2) \max (\|\psi(\cdot)\|_1, \|\dot{\psi}(\cdot)\|_1) + \|\hat{f}(\cdot)\|_1 \right) = \\
= K(\alpha, \delta, T)(1 + \tilde{K}h)^{\frac{N}{2}} \|\Phi(\cdot)\|_1, \quad i = 1, N + 1. \tag{2.9}
\]

Thus, the proof of the theorem is complete.
By functions \( v^{(i)}(t), \ i = 1, N + 1 \), we construct in \( \Omega \) the following functions
\[
U_h(x, t) = \psi(t) + h \sum_{j=0}^{i-1} v^{(j)}(t) + v^{(i)}(t)(x - (i - 1)h), \quad x \in [(i - 1)h, ih),
\]
\[
W_h(x, t) = \dot{\psi}(t) + h \sum_{j=0}^{i-1} \dot{v}^{(j)}(t) + \dot{v}^{(i)}(t)(x - (i - 1)h), \quad x \in [(i - 1)h, ih), \tag{2.10}
\]
\[
V_h(x, t) = v^{(i+1)}(t) \frac{x - (i - 1)h}{h} + v^{(i)}(t) \frac{ih - x}{h}, \quad x \in [(i - 1)h, ih), \quad i = 1, \bar{N}.
\]

The triple of functions \( \{U_h(x, t), W_h(x, t), V_h(x, t)\} \) is the approximate solution of the problem (2.1)-(2.3), constructed by modified Euler method.

4 The convergence of the modification of Euler method

In section 2 the scheme of modification of Euler method is given and the existence of a unique solution of the periodical boundary value problem for the system of ordinary differential equation (2.3), (2.4) is proved.

In this section we prove the convergence of modification of Euler method to the solution of boundary value problem (2.1)-(2.3), which is equivalent to the problem (1.3)-(1.3). We introduce the following notations

\[
\lambda(\omega, h) = [1 + K(\alpha, \delta, T)(\alpha_1 + \alpha_2) \cdot h]^\frac{\pi}{
\]
\[
\nu(\alpha, \delta, T) = \left(1 + \frac{e^\delta \cdot e^{\alpha T}}{e^{\delta} - 1}\right) \left[\frac{1}{\alpha} \left(T e^{\alpha T} + \frac{e^{\alpha T} - 1}{\alpha}\right) + \frac{e^{\alpha T} - 1}{\alpha} T \left(1 + \frac{e^{2\delta} \cdot e^{\alpha T}}{(e^{\delta} - 1)^2}\right)\right],
\]
\[
\hat{A} = \max (\|A(x, \cdot) - A((i - 1)h, \cdot)\|_1, \|A(x, \cdot) - A(\cdot, \cdot)\|_1),
\]
\[
\bar{B} = \max (\|B(x, \cdot) - B((i - 1)h, \cdot)\|_1, \|B(x, \cdot) - B(\cdot, \cdot)\|_1),
\]
\[
\tilde{C} = \max (\|\bar{C}(x, \cdot) - \bar{C}(x, \cdot)\|_1, \|\bar{C}(x, \cdot) - C(\cdot, \cdot)\|_1),
\]
\[
\|F(\cdot)\|_1 = \max (\|f(x, \cdot) - f((i - 1)h, \cdot)\|_1, \|f(x, \cdot) - f(\cdot, \cdot)\|_1),
\]
\[
\|\Phi(\cdot)\|_1 = (\alpha_1 + \alpha_2) \max (\|\dot{\Phi}(\cdot)\|_1, \|\dot{\Phi}(\cdot)\|_1) + \|\dot{\Phi}(\cdot)\|_1,
\]
\[
a_1^h(x) = \lambda(\omega, h) K(\alpha, \delta, T) \left(\alpha_1 + \alpha_2\right) + \left(\hat{A} + \bar{B} + \tilde{C}\right) \times \max (\|\dot{\Phi}(\cdot)\|_1, \|\dot{\Phi}(\cdot)\|_1),
\]
\[
a_2^h(x) = (1 + \lambda(\omega, h) K(\alpha, \delta, T) \left(\alpha_1 + \alpha_2\right) K(\alpha, \delta, T) \hat{A} \left(\dot{\Phi}(\cdot)\right)_{\|1},
\]
\[
A_1^h(x) = K_1(\alpha, \delta, T) (a_1^h(x) + \nu(\alpha, \delta, T) a_2^h(x)),
\]
\[
A_2^h(x) = \left(1 + \alpha K_1(\alpha, \delta, T) a_1^h(x) + (\alpha \nu(\alpha, \delta, T) + K_1(\alpha, \delta, T)) a_2^h(x),
\]
\[
The following statement establishes the estimate of proximity of the constructed triple of functions \( \{U_h(x, t), W_h(x, t), V_h(x, t)\} \) to the exact solution of the problem (1.1)-(1.3).

**Theorem 4.1** Let conditions of Theorem 1 hold. Then there exists a unique solution \( \{u^*(x, t), w^*(x, t), v^*(x, t)\} \) of the boundary value problem (1.1)-(1.3)
in $\Omega$ and it satisfies the following inequalities

$$\max \left( \left\| u^*(x, \cdot) - U_h(x, \cdot) \right\|_1, \left\| \frac{\partial u^*}{\partial t}(x, \cdot) - W_h(x, \cdot) \right\|_1 \right) \leq B_h^1(x) \exp \left( \int_0^x c(\xi) d\xi \right),$$

$$\left\| \frac{\partial u^*}{\partial x}(x, \cdot) - V_h(x, \cdot) \right\|_1 \leq A_h(x) + c(x)B_h^1(x) \exp \left( \int_0^x c(\xi) d\xi \right).$$

**Proof.** If the conditions of the theorem hold, then we show that there exists a solution of the boundary value problem (1.1)-(1.3). We take as the initial approximation $u = U_h(x, t)$, $w = W_h(x, t)$ and consider the following periodical boundary value problem

$$\frac{\partial v}{\partial t} = A(x, t)v + B(x, t)W_h(x, t) + C(x, t)U_h(x, t) + f(x, t), \quad (3.1)$$

$$v(x, 0) = v(x, T), \quad x \in [0, \omega], \quad (3.2)$$

For fixed value $x \in [0, \omega]$ the solution of the problem (3.1), (3.2) can be represented in the form

$$v^{(1)}(x, t) = \frac{\exp \left( \int_0^t A(x, \tau) d\tau \right)}{1 - \exp \left( \int_0^T A(x, \tau) d\tau \right)} \int_0^T F(x, \tau) \exp \left( \int_\tau^T A(x, \tau') d\tau' \right) d\tau +$$

$$+ \int_0^t F(x, \tau) \exp \left( \int_\tau^t A(x, \tau') d\tau' \right) d\tau, \quad (x, t) \in \Omega,$$

where $F(x, t) = B(x, t)W_h(x, t) + C(x, t)U_h(x, t) + f(x, t)$.

By virtue of representation (2.11), we have for the difference of $v^{(1)}(x, t)$ and $V_h(x, t)$

$$\left\| v^{(1)}(x, \cdot) - V_h(x, \cdot) \right\|_1 \leq$$

$$\leq \left\| v^{(1)}(x, \cdot) - v^{(i+1)}(\cdot) \right\|_1 \left\| \frac{x-(i-1)h}{h} + \frac{ih-x}{h} - v^{(i)}(\cdot) \right\|_1 \leq$$

$$\leq \left\| v^{(1)}(x, \cdot) - v^{(i+1)}(\cdot) \right\|_1 \frac{x-(i-1)h}{h} + \left\| v^{(1)}(x, \cdot) - v^{(i)}(\cdot) \right\|_1 \frac{ih-x}{h} \leq$$

$$\leq \max \left( \left\| v^{(1)}(x, \cdot) - v^{(i+1)}(\cdot) \right\|_1, \left\| v^{(1)}(x, \cdot) - v^{(i)}(\cdot) \right\|_1 \right). \quad (3.3)$$

Since, function $v^{(i)}(t)$ is the solution of boundary value problem (2.3), then the inequality holds

$$\left\| v^{(1)}(x, \cdot) - v^{(i)}(\cdot) \right\|_1 \leq$$

$$\leq \max_{t \in [0, T]} \left\| \frac{\exp \left( \int_0^t A(x, \tau) d\tau \right)}{1 - \exp \left( \int_0^T A(x, \tau) d\tau \right)} \int_0^T F(x, \tau) \exp \left( \int_\tau^T A(x, \tau) d\tau \right) d\tau +$$

$$+ \int_0^t F(x, \tau) \exp \left( \int_\tau^t A(x, \tau) d\tau \right) d\tau - \frac{\exp \left( \int_0^t A((i-1)h, \tau) d\tau \right)}{1 - \exp \left( \int_0^T A((i-1)h, \tau) d\tau \right)} \times$$
\[
\times \int_0^T F((i - 1)h, \tau) \exp \left( \int_\tau^T A((i - 1)h, \tau_1) d\tau_1 \right) d\tau - \int_0^t F((i - 1)h, \tau) \exp \left( \int_\tau^t A((i - 1)h, \tau_1) d\tau_1 \right) d\tau \leq \max_{t \in [0,T]} \left| \frac{\exp \left( \int_0^T A(x, \tau) d\tau \right)}{1 - \exp \left( \int_0^T A(x, \tau) d\tau \right)} \int_0^T \left\{ [F(x, \tau) - F((i - 1)h, \tau)] \times \right.
\exp \left( \int_\tau^T A(x, \tau_1) d\tau_1 \right) + \int_0^T F((i - 1)h, \tau) \left[ \exp \left( \int_\tau^T A(x, \tau_1) d\tau_1 \right) - \exp \left( \int_\tau^T A((i - 1)h, \tau_1) d\tau_1 \right) \right] d\tau + \int_0^t F((i - 1)h, \tau) \left[ \exp \left( \int_\tau^t A(x, \tau_1) d\tau_1 \right) - \exp \left( \int_\tau^t A((i - 1)h, \tau_1) d\tau_1 \right) \right] d\tau + \\
\left. \left[ \frac{\exp \left( \int_0^T A(x, \tau) d\tau \right)}{1 - \exp \left( \int_0^T A(x, \tau) d\tau \right)} - \frac{\exp \left( \int_0^T A((i - 1)h, \tau) d\tau \right)}{1 - \exp \left( \int_0^T A((i - 1)h, \tau) d\tau \right)} \right] \times \int_0^T F((i - 1)h, \tau) \exp \left( \int_\tau^T A((i - 1)h, \tau_1) d\tau_1 \right) d\tau \right| \leq \left( 1 + \frac{e^\delta \cdot e^{\alpha T} - 1}{\alpha} \right) \frac{\|F(x, \cdot) - F((i - 1)h, \cdot)\|_1 + \|e^\delta \cdot e^{\alpha T} - 1\|_1}{e^\delta - 1} \times \left\{ \frac{1}{\alpha} \left( T e^{\alpha T} + \frac{e^{\alpha T} - 1}{\alpha} \right) + \frac{e^{\alpha T} - 1}{\alpha} \left( 1 + \frac{e^{2\delta} \cdot e^{\alpha T}}{(e^\delta - 1)^2} \right) \right\} \|F((i - 1)h, \cdot)\|_1 \times \|A(x, \cdot) - A((i - 1)h, \cdot)\|_1 = K_1(\alpha, \delta, T) \|F(x, \cdot) - F((i - 1)h, \cdot)\|_1 + + \nu(\alpha, \delta, T) \|A(x, \cdot) - A((i - 1)h, \cdot)\|_1 \|F((i - 1)h, \cdot)\|_1, \right.
\left. x \in [(i - 1)h, ih]. \right. (3.4)
\]

Taking into account the structure of functions \(U_h(x, t), W_h(x, t)\), we establish the following estimates

\[
\|F(x, \cdot) - F((i - 1)h, \cdot)\|_1 = \|B(x, \cdot)W_h(x, \cdot) + C(x, \cdot)U_h(x, \cdot) + f(x, \cdot) - B((i - 1)h, \cdot)W_h((i - 1)h, \cdot) - C((i - 1)h, \cdot)U_h((i - 1)h, \cdot) - f((i - 1)h, \cdot)\|_1 \leq (\alpha_1 + \alpha_2) \max \left( \|U_h(x, \cdot) - U_h((i - 1)h, \cdot)\|_1, \right.
\left. \|W_h(x, \cdot) - W_h((i - 1)h, \cdot)\|_1 \right) + \|f(x, \cdot) - f((i - 1)h, \cdot)\|_1 + + \max \left( \|U_h((i - 1)h, \cdot)\|_1, \|W_h((i - 1)h, \cdot)\|_1 \right) \|C(x, \cdot) - C((i - 1)h, \cdot)\|_1 + + \|B(x, \cdot) - B((i - 1)h, \cdot)\|_1 \leq (\alpha_1 + \alpha_2) \left[ \max \left( \|\psi(\cdot)\|_1, \|\dot{\psi}(\cdot)\|_1 \right) + \omega[1 + K(\alpha, \delta, T) \times
\right.$$}
Thus, by substituting (3.5), (3.6) into (3.3), we obtain

\[
\|v(1)(x, \cdot) - \hat{v}(\cdot)\|_1 \leq (\alpha_1 + \alpha_2) \max (\|v(\cdot)\|_1, \|\hat{v}(\cdot)\|_1) + \|\hat{v}(\cdot)\|_1 \leq (\alpha_1 + \alpha_2) \max (\|v(\cdot)\|_1, \|\hat{v}(\cdot)\|_1) + \|\hat{v}(\cdot)\|_1 + \omega(\alpha_1 + \alpha_2) K(\alpha, \delta, T) \times [1 + K(\alpha, \delta, T)(\alpha_1 + \alpha_2) h] \hat{\parallel} (\alpha_1 + \alpha_2) \max (\|v(\cdot)\|_1, \|\hat{v}(\cdot)\|_1) + \|\hat{v}(\cdot)\|_1 = (1 + \lambda(\omega, h) \omega(\alpha_1 + \alpha_2) K(\alpha, \delta, T)) \|\Phi(\cdot)\|_1.
\]

By applying inequalities (3.4), we have

\[
\|v^{(i)}(x, \cdot) - v^{(i+1)}(\cdot)\|_1 \leq K_1(\alpha, \delta, T) \{\lambda(\omega, h) K(\alpha, \delta, T)(\alpha_1 + \alpha_2) h + (\hat{B} + \hat{C})\omega\} \|\Phi(\cdot)\|_1 + \|\hat{F}(\cdot)\|_1 + (\hat{B} + \hat{C}) \max (\|v(\cdot)\|_1, \|\hat{v}(\cdot)\|_1) + \nu(\alpha, \delta, T) \|A(x, \cdot) - A(\cdot)\|_1 (1 + \lambda(\omega, h) \omega(\alpha_1 + \alpha_2) K(\alpha, \delta, T)) \|\Phi(\cdot)\|_1, \quad x \in [(i - 1)h, ih), \quad i = 1, N. \quad (3.5)
\]

By arguing as in (3.5) for the difference \(v^{(i)}(x, t) - v^{(i+1)}(t)\), we have

\[
\|v^{(i)}(x, \cdot) - v^{(i+1)}(\cdot)\|_1 \leq K_1(\alpha, \delta, T) \{\lambda(\omega, h) K(\alpha, \delta, T) \times \left(\left(\alpha_1 + \alpha_2\right) h + (\hat{B} + \hat{C})\omega\right) \|\Phi(\cdot)\|_1 + \|\hat{F}(\cdot)\|_1 + (\hat{B} + \hat{C}) \max (\|v(\cdot)\|_1, \|\hat{v}(\cdot)\|_1) + \nu(\alpha, \delta, T) \|A(x, \cdot) - A(\cdot)\|_1 (1 + \lambda(\omega, h) \omega(\alpha_1 + \alpha_2) K(\alpha, \delta, T)) \|\Phi(\cdot)\|_1, \quad x \in [(i - 1)h, ih), \quad i = 1, N. \quad (3.6)
\]

Thus, by substituting (3.5), (3.6) into (3.3), we obtain

\[
\|v^{(1)}(x, \cdot) - V_h(x, \cdot)\|_1 \leq K_1(\alpha, \delta, T) \{\lambda(\omega, h) K(\alpha, \delta, T) \times \left(\left(\alpha_1 + \alpha_2\right) h + (\hat{B} + \hat{C})\omega\right) \|\Phi(\cdot)\|_1 + \|\hat{F}(\cdot)\|_1 + (\hat{B} + \hat{C}) \max (\|v(\cdot)\|_1, \|\hat{v}(\cdot)\|_1) + \nu(\alpha, \delta, T) \|A(x, \cdot) - A(\cdot)\|_1 (1 + \lambda(\omega, h) \omega(\alpha_1 + \alpha_2) K(\alpha, \delta, T)) \|\Phi(\cdot)\|_1 \} = A_1^h(x), \quad x \in [(i - 1)h, ih), \quad i = 1, N + 1. \quad (3.7)
\]

By equality (2.10), we have

\[
\|v^{(1)}_t(x, \cdot) - V_{ht}(x, \cdot)\|_1 \leq \|v^{(1)}_t(x, \cdot) - \hat{v}^{(i+1)}(\cdot)\|_1 \frac{|x - (i - 1)h|}{h} + \|v^{(1)}_t(x, \cdot) - \hat{v}^{(i)}(\cdot)\|_1 \times \left|\frac{ih - x}{h}\right| \leq \max (\|v^{(1)}_t(x, \cdot) - \hat{v}^{(i+1)}(\cdot)\|_1, \|v^{(1)}_t(x, \cdot) - \hat{v}^{(i)}(\cdot)\|_1). \quad (3.8)
\]

Since functions \(v^{(1)}(x, t)\) and \(v^{(i)}(t)\) satisfy the differential equations (3.1) and (2.3), then the following inequality holds

\[
\|v^{(1)}_t(x, \cdot) - \hat{v}^{(i)}(\cdot)\|_1 \leq 
\]
By substituting (2.9), (3.4) into the right hand side of (3.9), we obtain
\[\|(x, \cdot) - \hat{v}(\cdot)\|_1 \leq \alpha K_1(\alpha, \delta, T)\|F(x, \cdot) - F((i-1)h, \cdot)\|_1 + \nu(\alpha, \delta, T)\|A(x, \cdot) - A((i-1)h, \cdot)\|_1 \times \]
\[\times \|F((i-1)h, \cdot)\|_1 + \|A(x, \cdot) - A((i-1)h, \cdot)\|_1 K_1(\alpha, \delta, T)\|F((i-1)h, \cdot)\|_1 + \]
\[+ \|F(x, \cdot) - F((i-1)h, \cdot)\|_1 \leq (1 + \alpha K_1(\alpha, \delta, T))\left(\lambda(\omega, h)K(\alpha, \delta, T)\left((\alpha_1 + \alpha_2)h + (\hat{B} + \hat{C})\omega\right)\Psi(\cdot)\right) + \]
\[\|F(\cdot)\|_1 + (\hat{B} + \hat{C})\max(\|\psi(\cdot)\|_1, \|\hat{\psi}(\cdot)\|_1) + \alpha\nu(\alpha, \delta, T) + K_1(\alpha, \delta, T)\times \]
\[\times \|A(x, \cdot) - A((i-1)h, \cdot)\|_1 (1 + \lambda(\omega, h)\omega(\alpha_1 + \alpha_2)K(\alpha, \delta, T))\|\Phi(\cdot)\|_1. \] (3.10)
By arguing as in (3.10), we have
\[\|v_t^{(1)}(x, \cdot) - \hat{v}(\cdot)\|_1 \leq (1 + \alpha K_1(\alpha, \delta, T))\times \]
\[\times \left\{\lambda(\omega, h)K(\alpha, \delta, T)\left((\alpha_1 + \alpha_2)h + (\hat{B} + \hat{C})\omega\right)\Psi(\cdot)\right\} + \|F(\cdot)\|_1 + \]
\[+ (\hat{B} + \hat{C})\max(\|\psi(\cdot)\|_1, \|\hat{\psi}(\cdot)\|_1) + \alpha\nu(\alpha, \delta, T) + K_1(\alpha, \delta, T)\times \]
\[\times \|A(x, \cdot) - A((i-1)h, \cdot)\|_1 \left(1 + \lambda(\omega, h)\omega(\alpha_1 + \alpha_2)K(\alpha, \delta, T)\right)\|\Phi(\cdot)\|_1. \]
By applying the last inequality to the right-hand side of (3.8), we have
\[\|v_t^{(1)}(x, \cdot) - V_{ht}(x, \cdot)\|_1 \leq (1 + \alpha K_1(\alpha, \delta, T))\times \]
\[\times \left\{\lambda(\omega, h)K(\alpha, \delta, T)\left((\alpha_1 + \alpha_2)h + (\hat{B} + \hat{C})\omega\right)\Psi(\cdot)\right\} + \|F(\cdot)\|_1 + \]
\[+ (\hat{B} + \hat{C})\max(\|\psi(\cdot)\|_1, \|\hat{\psi}(\cdot)\|_1) + \alpha\nu(\alpha, \delta, T) + K_1(\alpha, \delta, T)\times \]
\[\times \left(1 + \lambda(\omega, h)\omega(\alpha_1 + \alpha_2)K(\alpha, \delta, T)\right)\|\Phi(\cdot)\|_1 = \tilde{A}_h^2(x). \] (3.11)
The inequalities (3.7) and (3.11) imply that
\[\max\left(\|v_t^{(1)}(x, \cdot) - V_{ht}(x, \cdot)\|_1, \|v_t^{(1)}(x, \cdot) - V_{ht}(x, \cdot)\|_1\right) \leq \]
\[\leq K(\alpha, \delta, T)\left\{\lambda(\omega, h)K(\alpha, \delta, T)\left((\alpha_1 + \alpha_2)h + (\hat{B} + \hat{C})\omega\right)\Psi(\cdot)\right\} + \|F(\cdot)\|_1 + \]
\[+ (\hat{B} + \hat{C})\max(\|\psi(\cdot)\|_1, \|\hat{\psi}(\cdot)\|_1) + \max(\nu(\alpha, \delta, T), \alpha\nu(\alpha, \delta, T) + K_1(\alpha, \delta, T))\times \]
\[\times \left(1 + \lambda(\omega, h)\omega(\alpha_1 + \alpha_2)K(\alpha, \delta, T)\right)\|\Phi(\cdot)\|_1 \leq \tilde{A}_h^2(x). \]
\[ x \hat{A} \cdot \left( 1 + \lambda(\omega, h)\omega(\alpha_1 + \alpha_2)K(\alpha, \delta, T) \right) \| \Phi(\cdot) \|_1 = A_h(x). \]  

(3.12)

By substituting \( v^{(1)}(x, t) \) and \( v_t^{(1)}(x, t) \) into functional expression (2.3), we have

\[ u^{(1)}(x, t) = \psi(t) + \int_0^x v^{(1)}(\xi, t) d\xi, \quad w^{(1)}(x, t) = \dot{\psi}(t) + \int_0^x v_t^{(1)}(\xi, t) d\xi. \]

By taking into account the structure of the functions \( U_h(x, t) \) and \( W_h(x, t) \), we obtain

\[ u^{(1)}(x, t) - U_h(x, t) = \int_0^x \left( v^{(1)}(\xi, t) - V_h(\xi, t) \right) d\xi + \int_0^x V_h(\xi, t) d\xi - \]

\[ -h \sum_{j=0}^{i-1} v^{(j)}(t) - v^{(i)}(t) [x - (i - 1)h], \]  

(3.13)

\[ w^{(1)}(x, t) - W_h(x, t) = \int_0^x \left( v_t^{(1)}(\xi, t) - V_{ht}(\xi, t) \right) d\xi + \int_0^x V_{ht}(\xi, t) d\xi - \]

\[ -h \sum_{j=0}^{i-1} \dot{v}^{(j)}(t) - \dot{v}^{(i)}(t) [x - (i - 1)h]. \]  

(3.14)

Further, (2.9) implies that

\[
\begin{align*}
\int_0^x V_h(\xi, t) d\xi &= \int_0^h \left[ v^{(1)}(t) \frac{\xi - h}{h} + v^{(2)}(t) \frac{\xi}{h} \right] d\xi + \int_h^{2h} \left[ v^{(2)}(t) \frac{2h - \xi}{h} + 
+ v^{(3)}(t) \frac{\xi - h}{h} \right] d\xi + \ldots + \int_{(i-2)h}^{(i-1)h} \left[ v^{(i-1)}(t) \frac{ih - \xi}{h} + v^{(i)}(t) \frac{\xi - (i - 1)h}{h} \right] d\xi + 
+ \int_{(i-1)h}^{x} \left[ v^{(i)}(t) \frac{ih - \xi}{h} + v^{(i+1)}(t) \frac{\xi - (i - 1)h}{h} \right] d\xi = \left[ v^{(1)}(t) + v^{(i)}(t) \right] \frac{h}{2} + 
+ h \left[ v^{(2)}(t) + v^{(2)}(t) + \ldots + v^{(i-1)}(t) \right] + \int_{(i-1)h}^{x} \left[ v^{(i)}(t) \frac{ih - \xi}{h} + 
+ v^{(i+1)}(t) \frac{\xi - (i - 1)h}{h} \right] d\xi.
\end{align*}
\]  

(3.15)

By substituting (3.13) into the right hand side of (3.15) and by virtue of inequalities (2.9), (3.7), we have

\[
\| u^{(1)}(x, \cdot) - U_h(x, \cdot) \|_1 \leq \int_0^x A_h^1(\xi) d\xi + \max \left( \| v^{(i)}(\cdot) \|_1, \| v^{(i+1)}(\cdot) \|_1 \right) |x - (i - 1)h| + 
+ \max \left( \| v^{(i)}(\cdot) \|_1, \| v^{(i)}(\cdot) \|_1 \right) h + \| v^{(i)}(\cdot) \|_1 |x - (i - 1)h| \leq \int_0^x A_h^1(\xi) d\xi + 
+ 3h \max_{i=1, N+1} \| v^{(i)}(\cdot) \|_1 \leq \int_0^x A_h^1(\xi) d\xi + 3hK(\alpha, \delta, T)\lambda(\omega, h)\| \Phi(\cdot) \|_1. \]

(3.16)
By arguing as in (3.16) and by (3.11), (3.14), we establish that

\[
\|w^{(1)}(x, \cdot) - W_h(x, \cdot)\|_1 \leq \int_0^x A_h^2(\xi) d\xi + \max (\|\dot{v}^{(i)}(\cdot)\|_1, \|\dot{v}^{(i+1)}(\cdot)\|_1)|x - (i-1)h| + \\
+ \max (\|\dot{v}^{(i)}(\cdot)\|_1, \|\dot{v}^{(i)}(\cdot)\|_1) \frac{h}{2} \|\dot{\varphi}(\cdot)\|_1|x - (i - 1)h| \leq \int_0^x A_h^2(\xi) d\xi + \\
+ 3h \max_{i=1,2,\ldots,N} \|\dot{v}(\cdot)\|_1 \leq \int_0^x A_h^2(\xi) d\xi + 3hK(\alpha, \delta, T)\lambda(\omega, h)\|\Phi(\cdot)\|_1. \tag{3.17}
\]

Inequalities (3.16) and (3.17) imply that

\[
\max (\|u^{(1)}(x, \cdot) - U_h(x, \cdot)\|_1, \|w^{(1)}(x, \cdot) - W_h(x, \cdot)\|_1) \leq 3hK(\alpha, \delta, T) \times \\
\times \lambda(\omega, h)\|\Phi(\cdot)\|_1 + \int_0^x \max (A_h^1(\xi) d\xi, A_h^2(\xi)) d\xi = B_h^1(x). \tag{3.18}
\]

We consider again the boundary value problem (2.1), (2.2) for \(u = u^{(1)}(x, t), w = w^{(1)}(x, t)\). Then it is easy to establish

\[
\|v^{(2)}(x, \cdot) - v^{(1)}(x, \cdot)\|_1 \leq K_1(\alpha, \delta, T)(\|B(x, \cdot)\|_1 + \|C(x, \cdot)\|_1) \times \\
\times \max (\|u^{(1)}(x, \cdot) - U_h(x, \cdot)\|_1, \|w^{(1)}(x, \cdot) - W_h(x, \cdot)\|_1) \leq \\
\leq K_1(\alpha, \delta, T)(\|B(x, \cdot)\|_1 + \|C(x, \cdot)\|_1)B_h^1(x), \tag{3.19}
\]

\[
\|v^{(2)}_t(x, \cdot) - v^{(1)}_t(x, \cdot)\|_1 \leq (1 + \alpha \cdot K_1(\alpha, \delta, T))(\|B(x, \cdot)\|_1 + \|C(x, \cdot)\|_1) \times \\
\times \max (\|u^{(1)}(x, \cdot) - U_h(x, \cdot)\|_1, \|w^{(1)}(x, \cdot) - W_h(x, \cdot)\|_1) \leq \\
\leq (1 + \alpha \cdot K_1(\alpha, \delta, T))(\|B(x, \cdot)\|_1 + \|C(x, \cdot)\|_1)B_h^1(x). \tag{3.20}
\]

Inequalities (3.19) and (3.20) imply that

\[
\max (\|u^{(2)}(x, \cdot) - u^{(1)}(x, \cdot)\|_1, \|v^{(2)}_t(x, \cdot) - v^{(1)}_t(x, \cdot)\|_1) \leq K(\alpha, \delta, T) \times \\
\times (\|B(x, \cdot)\|_1 + \|C(x, \cdot)\|_1) \max (\|u^{(1)}(x, \cdot) - U_h(x, \cdot)\|_1, \|w^{(1)}(x, \cdot) - W_h(x, \cdot)\|_1) \times \\
\leq K(\alpha, \delta, T)(\|B(x, \cdot)\|_1 + \|C(x, \cdot)\|_1)B_h^1(x) = c(x)B_h^1(x). \tag{3.21}
\]

The expression (2.3) implies that

\[
\max (\|u^{(2)}(x, \cdot) - u^{(1)}(x, \cdot)\|_1, \|w^{(2)}(x, \cdot) - w^{(1)}(x, \cdot)\|_1) \leq \\
\leq \int_0^x \max (\|u^{(2)}(\xi, \cdot) - u^{(1)}(\xi, \cdot)\|_1, \|v^{(2)}_t(\xi, \cdot) - v^{(1)}_t(\xi, \cdot)\|_1) d\xi \leq \\
\leq K(\alpha, \delta, T) \int_0^x (\|B(\xi, \cdot)\|_1 + \|C(\xi, \cdot)\|_1)B_h^1(\xi)d\xi = \int_0^x c(\xi)B_h^1(\xi) d\xi. \tag{3.22}
\]
Further, according to the algorithm we solve the boundary value problem (2.1), (2.2) for \( u \) following inequality hold. The functions (3.21), (3.22), we have

\[
\text{expression}
\]

Consequently, by estimating the differences \( w \) we can find the next approximation \( v \). By arguing as in (3.21), (3.22), we have

\[
\max \left( \| v^{(3)}(x, \cdot) - v^{(2)}(x, \cdot) \|_1 , \| v^i_t(x, \cdot) - v^i_t^{(2)}(x, \cdot) \|_1 \right) \leq K(\alpha, \delta, T)(\| B(x, \cdot) \|_1 + \| C(x, \cdot) \|_1) \max \left( \| u^{(2)}(x, \cdot) - u^{(1)}(x, \cdot) \|_1 , \| w^{(2)}(x, \cdot) - w^{(1)}(x, \cdot) \|_1 \right) \leq c(x) \int_0^x c(\xi) B_h^1(\xi) d\xi, \tag{3.23}
\]

\[
\max \left( \| u^{(3)}(x, \cdot) - u^{(2)}(x, \cdot) \|_1 , \| w^{(3)}(x, \cdot) - w^{(2)}(x, \cdot) \|_1 \right) \leq \int_0^x \max \left( \| v^{(3)}(\xi, \cdot) - v^{(2)}(\xi, \cdot) \|_1 , \| v^i_t(\xi, \cdot) - v^i_t^{(2)}(\xi, \cdot) \|_1 \right) d\xi \leq \int_0^x c(\xi) \int_0^{\xi} c(\xi_1) B_h^1(\xi_1) d\xi_1 d\xi = \frac{1}{2!} \left( \int_0^x c(\xi) d\xi \right)^2 B_h^1(x). \tag{3.24}
\]

Assuming that \( v^{(k)}(x, t), u^{(k)}(x, t), w^{(k)}(x, t), k = 2, 3, \ldots \), are known and the following inequality hold

\[
\max \left( \| v^{(k)}(x, \cdot) - v^{(k-1)}(x, \cdot) \|_1 , \| v^i_t(x, \cdot) - v^i_t^{(k-1)}(x, \cdot) \|_1 \right) \leq c(x) \times \int_0^x \max \left( \| v^{(k-1)}(\xi, \cdot) - v^{(k-2)}(\xi, \cdot) \|_1 , \| v^i_t^{(k-1)}(\xi, \cdot) - v^i_t^{(k-2)}(\xi, \cdot) \|_1 \right) d\xi, \tag{3.25}
\]

we can find the next approximation \( v \) by solving the family of periodical boundary value problem

\[
\frac{\partial v^{(k+1)}}{\partial t} = A(x, t) v^{(k+1)} + B(x, t) w^{(k)}(x, t) + C(x, t) u^{(k)}(x, t) + f(x, t), \tag{3.26}
\]

\[
v^{(k+1)}(x, 0) = v^{(k+1)}(x, T). \tag{3.27}
\]

The functions \( u^{(k+1)}(x, t), w^{(k+1)}(x, t) \) can be determined from functional expression

\[
u^{(k+1)}(x, t) = \psi(t) + \int_0^x v^{(k+1)}(\xi, t) d\xi,
\]

\[
w^{(k+1)}(x, t) = \dot{\psi}(t) + \int_0^x v^i_t^{(k+1)}(\xi, t) d\xi, \quad k = 1, 2, \ldots.
\]

Consequently, by estimating the differences \( u^{(k+1)}(x, t) - u^{(k)}(x, t) \) and \( w^{(k+1)}(x, t) - w^{(k)}(x, t) \), we obtain

\[
\max \left( \| u^{(k+1)}(x, \cdot) - u^{(k)}(x, \cdot) \|_1 , \| w^{(k+1)}(x, \cdot) - w^{(k)}(x, \cdot) \|_1 \right) \leq \int_0^x \max \left( \| v^{(k+1)}(\xi, \cdot) - v^{(k)}(\xi, \cdot) \|_1 , \| v^i_t^{(k+1)}(\xi, t) - v^i_t^{(k)}(\xi, t) \|_1 \right) d\xi. \tag{3.28}
\]
k = 1, 2, 3, \ldots, and by arguing as in (3.22), we have
\[
\max(\|v^{(k+1)}(x, \cdot) - v^{(k)}(x, \cdot)\|_1, \|v_t^{(k+1)}(x, \cdot) - v_t^{(k)}(x, \cdot)\|_1) \leq \\
\leq c(x) \int_0^x \max(\|v^{(k)}(\xi, \cdot) - v^{(k-1)}(\xi, \cdot)\|_1, \|v_t^{(k)}(\xi, \cdot) - v_t^{(k-1)}(\xi, \cdot)\|_1) d\xi \leq \\
\leq c(x) B^1_h(x) \frac{1}{(k - 2)!} \left( \int_0^x c(\xi) d\xi \right)^{k-2}, \quad k = 1, 2, 3, \ldots, \tag{3.29}
\]

\[
\max(\|v^{(k+1)}(x, \cdot) - v^{(k)}(x, \cdot)\|_1, \|v_t^{(k+1)}(x, \cdot) - v_t^{(k)}(x, \cdot)\|_1) \leq \\
\leq \max(\|v^{(k+1)}(x, \cdot) - v^{(k)}(x, \cdot)\|_1, \|v_t^{(k+1)}(x, \cdot) - v_t^{(k)}(x, \cdot)\|_1) + \ldots + \\
+ \max(\|v^{(2)}(x, \cdot) - v^{(1)}(x, \cdot)\|_1, \|v_t^{(2)}(x, \cdot) - v_t^{(1)}(x, \cdot)\|_1) + \\
+ \max(\|v^{(1)}(x, \cdot) - V_h(x, \cdot)\|_1, \|v_t^{(1)}(x, \cdot) - V_{ht}(x, \cdot)\|_1) \leq \\
\leq A_h(x) + B^1_h(x) \exp \left( \int_0^x c(\xi) d\xi \right), \tag{3.30}
\]

\[
\max(\|u^{(k+1)}(x, \cdot) - U_h(x, \cdot)\|_1, \|w^{(k+1)}(x, \cdot) - W_h(x, \cdot)\|_1) \leq \\
\leq \max(\|u^{(k+1)}(x, \cdot) - u^{(k)}(x, \cdot)\|_1, \|w^{(k+1)}(x, \cdot) - w^{(k)}(x, \cdot)\|_1) + \ldots + \\
+ \max(\|u^{(2)}(x, \cdot) - u^{(1)}(x, \cdot)\|_1, \|w^{(2)}(x, \cdot) - w^{(1)}(x, \cdot)\|_1) + \\
+ \max(\|u^{(1)}(x, \cdot) - U_h(x, \cdot)\|_1, \|w^{(1)}(x, \cdot) - W_h(x, \cdot)\|_1) \leq \\
\leq B^1_h(x) \exp \left( \int_0^x c(\xi) d\xi \right). \tag{3.31}
\]

By taking into account the uniformly continuity of function \(f(x, t)\) in \(\overline{\Omega}\) and the structure of functions \(A_h(x), B^1_h(x)\), as \(h \to 0\), the inequalities (3.28)-(3.31) imply that the sequence of functions \(\{u^{(k)}(x, t), w^{(k)}(x, t), v^{(k)}(x, t)\}, k = 1, 2, \ldots\) in \(\overline{\Omega}\) converges uniformly to the solution \(\{u^*(x, t), w^*(x, t), v^*(x, t)\}\) of the problem (2.1)-(2.3). Hence, it follows that semiperiodical boundary value problem (1.1)-(1.3) has the solution \(u^*(x, t)\). The uniqueness of the solution of boundary value problem (1.1)-(1.3) can be proved by contradiction. The proof of the theorem is complete.

5 Example

We consider the following boundary problem
\[
\frac{\partial^2 u}{\partial x \partial t} = t \cdot \sin(t - x) \frac{\partial u}{\partial x} + t \cdot \cos(t - x) u + \sin(t - x), \tag{4.1}
\]
\[
u(0, t) = sint, \quad t \in [0, 2\pi], \tag{4.2}
\]
\[
u(x, 0) = u(x, 2\pi), \quad x \in [0, 1]. \tag{4.3}
\]
in $\Omega = [0, 1] \times [0, 2\pi]$. The problem (4.1)-(4.3) has a solution, since the conditions of theorem 2 hold, i.e. $\left| \int_0^{2\pi} \tau \cdot \sin(\tau - x) d\tau \right| = 2\pi \cos(x) > 0$, $x \in [0, 1]$.

Then, by Theorem 2 the boundary value problem (4.1) - (4.2) has a solution. We convert the problem (4.1) - (4.2) into the equivalent problem (1.3) - (1.5) and apply the modification of Euler method. Then the approximate solution of the problem (4.1) - (4.2) has the following form

$$U_h(x, t) = \sin(t) + h \sum_{i=0}^{i-1} v^{(j)}(t) + v^{(i)}(t)(x - (i - 1)h), \quad x \in [0, (i - 1)h],$$

where

$$v^{(i)}(t) = \frac{\exp \left( \int_0^t \tau \cdot \sin(\tau - (i - 1)h) d\tau \right)}{1 - \exp \left( \int_0^{2\pi} \tau \cdot \sin(\tau - (i - 1)h) d\tau \right)} \int_0^t \left[ \tau \cdot \cos(\tau - (i - 1)h) \right] (\sin(\tau) +$$

$$+ h \sum_{j=0}^{i-1} v^{(j)}(\tau)) + \sin(\tau - (i - 1)h)] \exp \left( \int_\tau^{2\pi} \tau_1 \cdot \sin(\tau_1 - (i - 1)h) d\tau_1 \right) d\tau +$$

$$+ \int_\tau^{2\pi} \tau \cdot \cos(\tau - (i - 1)h) \left( \sin(\tau) + h \sum_{j=0}^{i-1} v^{(j)}(\tau) \right) + \sin(\tau - (i - 1)h) \times$$

$$\times \exp \left( \int_0^\tau \tau_1 \cdot \sin(\tau_1 - (i - 1)h) d\tau_1 \right) d\tau.$$

(4.4) implies that $U_h(x, t)$ approaches to the exact solution of the problem (4.1)-(4.2) as $h \to 0$.

**Conclusion**

In this paper on the basis of the modification of Euler method an algorithm for finding an approximate solution of semiperiodical boundary value problem for a linear hyperbolic equations with mixed derivative (1)-(3) is constructed. The equivalence of problems (1)-(3) and (1.1)-(1.3) is proved. The area considered is divided into $N$ parts with respect to $x$ and a family of periodic boundary value problems for ordinary differential equations (2.2), (2.3) is obtained. The conditions of existence and uniqueness of the solution of the problem (2.2), (2.3) are established. By solution of (2.2), (2.3) the triple of functions (2.10) is constructed and the convergence of functions $\{U_h(x, t), W_h(x, t), V_h(x, t)\}$ to the solution and its derivatives with respect to $t$ and $x$, respectively, are proved.

**Conflict of interest.** The authors declare that they have no competing interests.

**Authors contributions.** Both authors contributed to the writing of this article equally. Both authors read and approved the final manuscript.
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