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Analytical Method for Finding Polynomial Roots

V. V. Kamyshlov[*]

Institute of Mathematical Problems of Biology, Russian Academy of Science
Institutskaya street, 4, Pushchino, 142290, Moscow region, Russia
[*]Coresponding author

V. S. Bystrov

Institute of Mathematical Problems of Biology, Russian Academy of Science
Institutskaya street, 4, Pushchino, 142290, Moscow region, Russia

Abstract

We present a method to get an analytical expression for finding polynomial roots. The roots are expressed as infinite power series. The terms of the series are written as nearly triangular determinants. The structure of the series and formulas for finding the coefficients are given. By way of illustration we derive an analytical solution of a third degree polynomial. The method can be used to find the roots of some other algebraic equations as well. This method of solving polynomials is a part of a more general research task concerned with finding solutions of nonlinear problems.

Keywords: nonlinear algebraic equations, polynomials, roots of polynomials, analytical solution, Newton's method, algebraic functions

Introduction

For many years we have witnessed interest of researchers in finding an analytical solution of polynomials. With the use of formulae, i.e. via a finite number of algebraic operations second-, third- and fourth-degree equations were solved. Then Abel-Ruffini theorem was proved which holds that fifth- and higher-degree equations have no solution in the form of a formula. This theorem put a period to longstanding search for a 'magic' formula. However the theorem said nothing of whether polynomials can be solved with infinite power series or

not. Many attempts to do so were taken, some authors succeeded [1]-[5], but a general method was not developed. An important role in finding polynomial roots has belonged to numerical approaches. One of them is iteration Newton's method. It provides good convergence and enables one to find solutions of nonlinear equations. The method is fully considered in [6]-[10]. When dealing with iteration Newton's method, it turned out that it can be transformed from iteration to analytical one. This problem was partly solved in [11] in 1951.
Preassigned direct function

$$y = y_0 + \sum_{n=1}^{\infty} a_n (x - x_0)^n, \ a_1 \neq 0. \tag{1}$$

Unknown inverse function

$$x = x_0 + \sum_{n=1}^{\infty} b_n (y - y_0)^n. \tag{2}$$

By replacing the variable, the problem is easily reduced to the case

$$x_0 = 0, \quad y_0 = 0.$$

Then the coefficients of the inverse function will be expressed in terms of nearly triangular determinants as

$$b_1 = \frac{1}{a_1}; \quad b_2 = -\frac{a_2}{a_1^3}; \quad b_3 = \frac{1}{3!a_1^5}\begin{vmatrix} 3a_2 & a_1 \\ 6a_3 & 4a_2 \end{vmatrix}; \quad \cdots$$

$$b_n = \frac{(-1)^{n-1}}{n!a_1^{2n-1}}\begin{vmatrix} na_2 & a_1 & 0 & 0 & \cdots \\ 2na_3 & (n+1)a_2 & 2a_1 & 0 & \cdots \\ 3na_4 & (2n+1)a_3 & (n+2)a_2 & 3a_1 & \cdots \\ 4na_5 & (3n+1)a_4 & (2n+2)a_3 & (n+3)a_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}.$$

But this is an incomplete solution of the problem. Since expansions of the form of (1) take place only in the vicinity of regular points, then the expansion of the inverse function (2) will be in the vicinity of regular points too. But since the inverse function has singular points, then convergence of series (3) will be limited. This raises the problem of developing an analytic continuation. It is essential to get expansions of the type of (2) in the vicinity of singular points. Algebraic functions have singular points of two types: algebraic critical points and poles of finite order. Having determined regularities for the coefficients of the series which represent an inverse function in the vicinity of these points we can get all the series which represent an inverse function on the whole complex plane. In this way we can solve the problem of finding polynomial roots for any values of coefficients belonging to the field of complex numbers.

1. Expansion into Newton's series in the vicinity of regular points

Newton's method is a numerical approach which is used to find the roots of nonlinear equations and systems of nonlinear equations.

Let a be a root of a doubly differentiable function and x_0 be a close approximation to the root. Then the following approximate equality

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0), \tag{1.1}$$

holds for all x, which are sufficiently close to x_0. Assuming $x = a$, we have

$$x_1 = a \approx x_0 - \frac{f(x_0)}{f'(x_0)}, \tag{1.2}$$

where x_1 is the first approximation of the root. If we put $x_0 = x_1$ and calculate (1.2) once again we will get the second approximation of the root – x_2, etc. Having performed iteration (1.2) as many times as necessary we will get the value of the root to the specified accuracy. Formula (1.2) is the beginning of the Taylor series for function *f(x)* in the vicinity of point x_0. Let us find higher expansion terms of the Taylor series in the vicinity of this point.

Let, in the field of complex numbers, a polynomial function of arbitrary degree $\omega = L(z)$ be specified. A function inverse to $\omega = L(z)$ which is multivalued in the general case, will be denoted as $z = K(\omega)$. Let z be a simple root of the algebraic expression $L(z) = 0, L'(z) \neq 0$.

Let us take a point z_0 in the vicinity of the point z and expand the function $z = K(\omega)$ into Taylor series. This expansion has the form

$$z - z_0 = \frac{\partial z(z_0)}{\partial \omega}(\omega - \omega_0) + \frac{1}{2!}\frac{\partial^2 z(z_0)}{\partial \omega^2}(\omega - \omega_0)^2 + \\ + \frac{1}{3!}\frac{\partial^3 z(z_0)}{\partial \omega^3}(\omega - \omega_0)^3 + \dots . \tag{1.3}$$

Let us find the value of $z - z_0$ in the first approximation. To do this we will drop all the terms of (1.3) of order 2 and higher. We will get

$$z - z_0 = \frac{\partial z(z_0)}{\partial \omega}(\omega - \omega_0) + \dots . \tag{1.4}$$

Having regard to the fact that $\omega(z) = 0$, since z is the root of the equation and taking into account that $\omega_0 = L(z_0)$, expression (1.4) takes on the form

$$z_1 - z_0 = \frac{\partial z(z_0)}{\partial \omega}(0 - L(z_0)) + \ldots . \tag{1.5}$$

Bearing in mind that

$$\frac{\partial z(z_0)}{\partial \omega} = \frac{1}{\frac{\partial \omega(z_0)}{\partial z}} = \frac{1}{L'(z_0)},$$

finally, for the first approximation of z, we get

$$z_1 = z_0 - \frac{L(z_0)}{L'(z_0)}. \tag{1.6}$$

Let us pass on to finding the second approximation of z

$$z_2 = z_0 - \frac{L(z_0)}{L'(z_0)} + \frac{\partial z(z_0)}{\partial \omega}(\omega - \omega_1) + \ldots . \tag{1.7}$$

Taking into account, as in the previous calculation, that $\omega(z) = 0$ and $\omega_1 = L(z_1)$, we obtain

$$z_2 = z_0 - \frac{L(z_0)}{L'(z_0)} + \frac{\partial z(z_0)}{\partial \omega}(0 - L(z_1)) + \ldots . \tag{1.8}$$

Let us expand $\omega_1 = L(z_1)$ in terms of $\frac{L(z_0)}{L'(z_0)}$

$$\omega_1 = L(z_1) = L\left[z_0 - \frac{L(z_0)}{L'(z_0)}\right] =$$
$$= L(z_0) + \frac{L'(z_0)}{1!}\left[-\frac{L(z_0)}{L'(z_0)}\right] + \frac{L''(z_0)}{2!}\left[-\frac{L(z_0)}{L'(z_0)}\right]^2 +$$
$$+ \frac{L'''(z_0)}{3!}\left[-\frac{L(z_0)}{L'(z_0)}\right]^3 + \ldots . \tag{1.9}$$

In (1.9) we will neglect the terms of the third order and higher and substitute this expression in (1.8).

$$z_2 = z_0 - \frac{L(z_0)}{L'(z_0)} + \frac{\partial z(z_0)}{\partial \omega}\begin{bmatrix} 0 - L(z_0) + L(z_0) - \\ -\frac{L''(z_0)}{2!}\left(-\frac{L(z_0)}{L'(z_0)}\right)^2 - \\ -\dots \end{bmatrix},$$

finally we get the second approximation

$$z_2 = z_0 - \frac{L(z_0)}{L'(z_0)} - \frac{L''(z_0)}{2!}\left(-\frac{[L(z_0)]^2}{[L'(z_0)]^3}\right). \tag{1.10}$$

Now let us pass on to finding the third approximation z_3.

$$\omega_2 = L(z_2) = L\left[z_0 - \frac{L(z_0)}{L'(z_0)} - \frac{L''(z_0)}{2!}\left(-\frac{[L(z_0)]^2}{[L'(z_0)]^3}\right)\right]. \tag{1.11}$$

Expanding (1.11) into series in terms of $\left[-\frac{L(z_0)}{L'(z_0)} - \frac{L''(z_0)}{2!}\left(-\frac{[L(z_0)]^2}{[L'(z_0)]^3}\right)\right]$.

$$\omega_2 = L(z_2) = L\left[z_0 - \frac{L(z_0)}{L'(z_0)} - \frac{L''(z_0)}{2!}\left(-\frac{[L(z_0)]^2}{[L'(z_0)]^3}\right)\right] =$$

$$= L(z_0) + L'(z_0)\left[-\frac{L(z_0)}{L'(z_0)} - \frac{L''(z_0)}{2!}\left(-\frac{[L(z_0)]^2}{[L'(z_0)]^3}\right)\right] +$$

$$+\frac{L''(z_0)}{2!}\left[-\frac{L(z_0)}{L'(z_0)} - \frac{L''(z_0)}{2!}\left(-\frac{[L(z_0)]^2}{[L'(z_0)]^3}\right)\right]^2 +$$

$$+\frac{L'''(z_0)}{3!}\left[-\frac{L(z_0)}{L'(z_0)} - \frac{L''(z_0)}{2!}\left(-\frac{[L(z_0)]^2}{[L'(z_0)]^3}\right)\right]^3 + \dots,$$

with regard to the fact (as in the previous calculations) that $\omega(z) = 0$ and $\omega_2 = L(z_2)$,and neglecting the terms of the fourth order and higher we will have

$$z_3 = z_0 - \frac{L(z_0)}{L'(z_0)} - \frac{L''(z_0)}{2!}\frac{\left[L(z_0)\right]^2}{\left[L'(z_0)\right]^3} +$$
$$+\left[\frac{L'''(z_0)}{6L'(z_0)^4} - \frac{L''(z_0)^2}{2L'(z_0)^5}\right]\left[L(z_0)\right]^3 + \ldots. \tag{1.12}$$

Similarly we will get the fourth approximation, z_4 .

$$z_4 = z_0 - \frac{L(z_0)}{L'(z_0)} - \frac{L''(z_0)}{2!}\frac{\left[L(z_0)\right]^2}{\left[L'(z_0)\right]^3} +$$
$$+\left[\frac{L'''(z_0)}{6L'(z_0)^4} - \frac{L''(z_0)^2}{2L'(z_0)^5}\right]\left[L(z_0)\right]^3 +$$
$$+\frac{L(z_0)^4}{4!L'(z_0)^7}\left[\begin{array}{l}15L''(z_0)^3 - 10L'(z_0)L''(z_0)L'''(z_0) + \\ +L'(z_0)^2 L''''(z_0)\end{array}\right] + \ldots. \tag{1.13}$$

In this way we can obtain any number of approximations, though the expressions which arise in higher approximations are rather complicated. It turned out that the terms of series (1.13) can be expressed via nearly triangular determinants. Let us rewrite expression (1.13) in a somewhat different form

$$z_4 = z_0 - \frac{L(z_0)}{L'(z_0)} - \frac{L(z_0)^2}{L'(z_0)^3}\left[\frac{L''(z_0)}{2!}\right] -$$
$$-\frac{L(z_0)^3}{2!L'(z_0)^5}\left[L''(z_0)^2 - \frac{L'(z_0)L'''(z_0)}{3}\right] -$$
$$-\frac{L(z_0)^4}{3!L'(z_0)^7}\left[\frac{10L'(z_0)L''(z_0)L'''(z_0) - 15L''(z_0)^3 - L'(z_0)^2 L''''(z_0)}{4}\right] -$$
$$-\ldots. \tag{1.14}$$

Expressions in square brackets can be presented in the form of nearly triangular determinants as

$$\Delta_2 = \left[\frac{L''(z_0)}{2!} \right] = \frac{1L''}{2!};$$

$$\Delta_3 = \left[L''(z_0)^2 - \frac{L'(z_0)L'''(z_0)}{3} \right] = \begin{vmatrix} \frac{1L''}{2!} & \frac{2L'''}{3!} \\ \frac{L'}{1!} & \frac{4L''}{2!} \end{vmatrix};$$

$$\Delta_4 = \left[\frac{10L'(z_0)L''(z_0)L'''(z_0) - 15L''(z_0)^3 - L'(z_0)^2 L''''(z_0)}{4} \right] = \tag{1.15}$$

$$= \begin{vmatrix} \frac{1L''}{2!} & \frac{2L'''}{3!} & \frac{3L''''}{4!} \\ \frac{L'}{1!} & \frac{5L''}{2!} & \frac{9L'''}{3!} \\ 0 & \frac{2L'}{1!} & \frac{6L''}{2!} \end{vmatrix}.$$

The determinant number is one greater than its dimensionality. This is done to simplify the form of expressions. In expressions (1.15) a regularity is noticeable which can be used to write down and then calculate all the other terms of the setries. The fourth coefficient will be presented as

$$a_4 = \frac{L^5}{4!(L')^9}\Delta_5; \quad \Delta_5 = \begin{vmatrix} \frac{1L''}{2!} & \frac{2L'''}{3!} & \frac{3L''''}{4!} & \frac{4L'''''}{4!} \\ \frac{L'}{1!} & \frac{6L''}{2!} & \frac{11L'''}{3!} & \frac{16L''''}{3!} \\ 0 & \frac{2L'}{1!} & \frac{7L''}{2!} & \frac{12L'''}{3!} \\ 0 & 0 & \frac{3L'}{1!} & \frac{8L''}{2!} \end{vmatrix}. \tag{1.16}$$

In the general form nearly triangular determinants are written as

$$\Delta_{n-1} =$$

$$\begin{array}{cccccccc}
\frac{1L''}{2!} & \frac{2L'''}{3!} & \frac{3L''''}{4!} & \frac{4L'''''}{5!} & \cdots & \cdots & \frac{(n-2)L^{n-1}}{(n-1)!} & \frac{(n-1)L^{n}}{(n)!} \\
\frac{L'}{1!} & \frac{(n+1)L''}{2!} & \frac{(2n+1)L'''}{3!} & \frac{(3n+1)L''''}{4!} & \cdots & \cdots & \frac{((n-2)n+1)L^{n-2}}{(n-2)!} & \frac{((n-2)n+1)L^{n-1}}{(n-1)!} \\
0 & \frac{2L'}{1!} & \frac{(n+2)L''}{2!} & \frac{(2n+2)L'''}{3!} & \cdots & \cdots & \frac{((n-4)n+2)L^{n-3}}{(n-3)!} & \frac{((n-3)n+2)L^{n-2}}{(n-2)!} \\
0 & 0 & \frac{3L'}{1!} & \frac{(n+3)L''}{2!} & \cdots & \cdots & \frac{((n-5)n+3)L^{n-4}}{(n-4)!} & \frac{((n-4)n+3)L^{n-3}}{(n-3)!} \\
 & & & & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \frac{(n-4)L'}{1!} & \frac{(2n-4)L''}{2!} & \frac{(3n-4)L'''}{3!} & \frac{(4n-4)L''''}{4!} \\
0 & 0 & 0 & \vdots & 0 & \frac{(n-3)L'}{1!} & \frac{(2n-3)L''}{2!} & \frac{(3n-3)L'''}{3!} \\
0 & 0 & 0 & \vdots & 0 & 0 & \frac{(n-2)L'}{1!} & \frac{(2n-2)L''}{2!} \\
0 & 0 & 0 & \vdots & & & &
\end{array}$$

(1.17)

As is easy to see, determinant (1.17) has the top right part, the main diagonal and one subdiagonal, the remaining bottom part is filled with zeros. Expanding such a determinant we can find the value of any coefficient. In the general form expansion in the vicinity of a regular point will be

$$z = z_0 - \frac{L(z_0)}{L'(z_0)} - \sum_{n=2}^{\infty} \frac{L(z_0)^n}{(n-1)!L'(z_0)^{2n-1}} \Delta_n(z_0) \ , \tag{1.18}$$

where Δ_n is a nearly triangular determinant which is found from expression (1.17).

2. Series expansion in the vicinity of a finite-order pole

Let us consider expansion of the inverse function $z = K(\omega)$ in the vicinity of the point at infinity which, for a *m*-th degree polynomial is a pole of the *m*-th order. For the inverse function $z = K(\omega)$, in the vicinity of the point at infinity we have the following expansion

$$\begin{aligned}
z - z_0 &= c_{-1}\left[L(z)-L(z_0)\right]^{\frac{1}{m}} + c_0\left[L(z)-L(z_0)\right]^{\frac{0}{m}} + \\
&+ c_1\left[L(z)-L(z_0)\right]^{-\frac{1}{m}} + \\
&+ c_2\left[L(z)-L(z_0)\right]^{-\frac{2}{m}} + c_3\left[L(z)-L(z_0)\right]^{-\frac{3}{m}} + \\
&+ c_4\left[L(z)-L(z_0)\right]^{-\frac{4}{m}} + \ldots .
\end{aligned} \tag{2.1}$$

In this expansion we should determine the coefficients $c_{-1}, c_0, c_1, c_2, c_3, \ldots$
Let us make the following replacement

$$L(z)-L(z_0)=z^m+p_1z^{m-1}+p_2z^{m-2}+\ldots-z_0^m-p_1z_0^{m-1}-$$
$$-p_2z_0^{m-2}-\ldots\approx z^m-z_0^m\approx(z-z_0)^m,$$
$$z_0=\infty,$$
$$z\to\infty. \tag{2.2}$$

Then expression (2.1) takes the form

$$z-z_0=c_{-1}\left[L(z-z_0)\right]^{\frac{1}{m}}+c_0\left[L(z-z_0)\right]^{\frac{0}{m}}+c_1\left[L(z-z_0)\right]^{-\frac{1}{m}}+$$
$$+c_2\left[L(z-z_0)\right]^{-\frac{2}{m}}+c_3\left[L(z-z_0)\right]^{-\frac{3}{m}}+c_4\left[L(z-z_0)\right]^{-\frac{4}{m}}+\ldots. \tag{2.3}$$

In the first approximation we will suppose that all the coefficients, except for c_{-1}, are equal to zero and the terms $(z-z_0)$ with powers less than the *m*-th one can be neglected. Then we get

$$z-z_0=c_{-1}\left[L(z-z_0)\right]^{\frac{1}{m}}=$$
$$=c_{-1}((z-z_0)^m+\frac{L^{m-1}}{(m-1)!}(z-z_0)^{m-1}+\frac{L^{m-2}}{(m-2)!}(z-z_0)^{m-2}+\ldots)^{\frac{1}{m}}= \tag{2.4}$$
$$=c_{-1}((z-z_0)^m)^{\frac{1}{m}}.$$

From (2.4) we find the value of c_{-1} to be
$c_{-1}=1.$
The second approximation will have the form

$$z-z_0=c_{-1}\left[L(z-z_0)\right]^{\frac{1}{m}}+c_0=$$
$$=1\ ((z-z_0)^m+\frac{L^{m-1}}{(m-1)!}(z-z_0)^{m-1}+\frac{L^{m-2}}{(m-2)!}(z-z_0)^{m-2}+\ldots)^{\frac{1}{m}}+ \tag{2.5}$$
$$+c_0=(z-z_0)\left(1+\frac{L^{m-1}}{(m-1)!}\frac{1}{z-z_0}+\ldots\right)^{\frac{1}{m}}+c_0\ .$$

In (2.5), in the second approximation, we suppose that all the coefficients, except for c_{-1} and c_0, are equal to zero and the terms $(z-z_0)$ with the powers less than the *(m-1)*-th one can be neglected. After simple transformations, the second approximation will be

$$1=\left(1+\frac{L^{m-1}}{(m-1)!}\frac{1}{z-z_0}+\ldots\right)^{\frac{1}{m}}+\frac{c_0}{z-z_0}. \tag{2.6}$$

The expression in the parentheses will be expanded into infinite series. In the resultant formula, we neglect the high-order terms and obtain the following approximate equality

$$\begin{aligned} 1&=1+\frac{L^{m-1}}{(m-1)!}\frac{1}{z-z_0}\frac{1}{m}+\ldots+\frac{c_0}{z-z_0};\\ 0&=\frac{L^{m-1}}{(m-1)!}\frac{1}{m}+c_0. \end{aligned} \tag{2.7}$$

From (2.7) we get the value of c_0 to be

$$c_0=-\frac{L^{m-1}}{(m-1)!}\frac{1}{m}. \tag{2.8}$$

The third coefficient is found similarly to the first two ones

$$\begin{aligned} z-z_0&=(z-z_0)\left[1+\frac{L^{m-1}}{(m-1)!}(z-z_0)^{-1}+\frac{L^{m-2}}{(m-2)!}(z-z_0)^{-2}+\ldots\right]^{\frac{1}{m}}+\\ &+\left[-\frac{L^{m-1}}{(m-1)!m}\right]+(z-z_0)\left[\begin{matrix}1+\dfrac{L^{m-1}}{(m-1)!}(z-z_0)^{-1}+\\ +\dfrac{L^{m-2}}{(m-2)!}(z-z_0)^{-2}+\ldots\end{matrix}\right]^{-\frac{1}{m}}c_1. \end{aligned} \tag{2.9}$$

In equality (2.9), we expand the expressions into brackets into series

$$z - z_0 = (z - z_0)\begin{bmatrix} 1 + \frac{1}{m}\frac{L^{m-1}}{(m-1)!}(z-z_0)^{-1} + \\ + \frac{1}{m}\frac{L^{m-2}}{(m-2)!}(z-z_0)^{-2} + \\ + \frac{\frac{1}{m}(\frac{1}{m}-1)}{2!}\left[\frac{L^{m-1}}{(m-1)!}\right]^2 (z-z_0)^{-2} + \\ + \frac{\frac{1}{m}(\frac{1}{m}-1)}{2!}\frac{2L^{m-1}}{(m-1)!}\frac{L^{m-2}}{(m-2)!}(z-z_0)^{-3} + \\ + \frac{\frac{1}{m}(\frac{1}{m}-1)}{2!}\left[\frac{L^{m-2}}{(m-2)!}\right]^2 (z-z_0)^{-4} \end{bmatrix} +$$

$$+\left[-\frac{L^{m-1}}{(m-1)!m}\right]+$$

$$+ \quad (z-z_0)^{-1}\begin{bmatrix} 1 - \frac{1}{m}\frac{L^{m-1}}{(m-1)!}(z-z_0)^{-1} - \\ - \frac{1}{m}\frac{L^{m-2}}{(m-2)!}(z-z_0)^{-2} + \\ + \frac{-\frac{1}{m}(-\frac{1}{m}-1)}{2!}\left[\frac{L^{m-1}}{(m-1)!}\right]^2 (z-z_0)^{-2} + \\ + \frac{-\frac{1}{m}(-\frac{1}{m}-1)}{2!}\frac{2L^{m-1}}{(m-1)!}\frac{L^{m-2}}{(m-2)!}(z-z_0)^{-3} + \\ + \frac{-\frac{1}{m}(-\frac{1}{m}-1)}{2!}\left[\frac{L^{m-2}}{(m-2)!}\right]^2 (z-z_0)^{-4} \end{bmatrix} c_1 . \tag{2.10}$$

In equality (2.10), we make equal the coefficients of $(z - z_0)^{-1}$

$$0=\frac{1}{m}\frac{L^{m-2}}{(m-2)!}+\frac{\frac{1}{m}\left(\frac{1-m}{m}\right)}{2!}\left[\frac{L^{m-1}}{(m-1)!}\right]^2+c_1;$$

$$c_1=-\frac{1}{2!m^2}\left[\frac{2mL^{m-2}}{(m-2)!}+(1-m)\left[\frac{L^{m-1}}{(m-1)!}\right]^2\right]. \tag{2.11}$$

Using this line of reasoning and carrying out expansions similar to those described above we can get the other coefficients as well. In the general form coefficients $c_{-1}, c_0, c_1, \ldots, c_n$ can be presented in terms of nearly triangular determinants as

$$c_{-1}=1;$$
$$c_{n-1}=-\frac{\Delta_n^*(z_0)}{n!m^n}. \tag{2.12}$$

where

Δ_n^*- is a nearly triangular determinant of the form

$$\Delta_n^*=$$

$$\begin{vmatrix}
\frac{1L^{(m-1)}}{(m-1)!} & \frac{2L^{(m-2)}}{(m-2)!} & \cdots & \frac{(m-1)L'}{1!} & 0 & \cdots & 0 & 0 \\
-m & \frac{((n-1)-m)L^{m-1}}{(m-1)!} & \cdots & \frac{((m-2)(n-1)-m)L''}{2!} & \frac{((m-1)(n-1)-m)L'}{1!} & 0 & \cdots & 0 \\
0 & -2m & \cdots & \frac{((m-3)(n-1)-2m)L'''}{3!} & \frac{((m-2)(n-1)-2m)L''}{2!} & \frac{((m-1)(n-1)-2m)L'}{1!} & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & \cdots & \cdots & \cdots & \cdots\ \cdots & \frac{((n-1)(m-2)-(n+1-m))L''}{2!} & \cdots & \frac{((n-1)(m-1)-(n+1-m))L'}{1!} \\
0 & 0 & 0 & \vdots & \cdots\ \cdots & \cdots & & \cdots \\
0 & 0 & 0 & \vdots & 0 \quad -(n-2)m & \frac{((n-1)-(n-2)m)L^{m-1}}{(m-1)!} & & \frac{(2(n-1)-(n-2)m)L^{m-2}}{(m-2)!} \\
0 & 0 & 0 & \vdots & 0 \quad 0 & -(n-1)m & & \frac{((n-1)-(n-1)m)L^{m-1}}{(m-1)!}
\end{vmatrix} \tag{2.13}$$

Accordingly, series (2.1) will be written as

$$z=z_0+\sqrt[m]{-L(z_0)}-\sum_{n=1}^{\infty}\frac{\Delta_n^*(z_0)}{n!m^n}\left[\sqrt[m]{-L(z_0)}\right]^{-(n-1)}. \tag{2.14}$$

Thus, we have series expansion for function in the vicinity of a regular point and in the vicinity of a finite-order pole. For algebraic functions, there remains one more type of points, namely, algebraic branch points. In the next section we will find expansions for these points too.

3. Series expansion in the vicinity of an algebraic branch point

Let us consider expansion of the inverse function $z = K(\omega)$ in the vicinity of an algebraic branch point of the κ-th order. In the vicinity of this point we have the following expansion

$$z = z_0 + \sum_{n=1}^{\infty} b_n \left[L(z) - L(z_0)\right]^{\frac{n}{k}} . \qquad (3.1)$$

The condition of availability of an algebraic branch point of the *κ-th* order is

$$L'(z_0) = L''(z_0) = \ldots = L^{k-1}(z_0) = 0 , \qquad (3.2)$$
$$L^k(z_0) \neq 0.$$

Let us write down series (3.1) in greater detail

$$z - z_0 = b_1 \left[L(z) - L(z_0)\right]^{\frac{1}{k}} + b_2 \left[L(z) - L(z_0)\right]^{\frac{2}{k}} + \qquad (3.3)$$
$$+ b_3 \left[L(z) - L(z_0)\right]^{\frac{3}{k}} + \cdots .$$

Our problem is to find coefficients $b_1, b_2, \ldots, b_n$ in expansion (3.3). With regard to (3.2), polynomial $L(z)$ in the vicinity of an algebraic branch point will have the form

$$L(z - z_0) = \frac{L^k(z_0)}{k!}(z - z_0)^k + \frac{L^{k+1}(z_0)}{(k+1)!}(z - z_0)^{k+1} +$$
$$+ \cdots + \frac{L^m(z_0)}{m!}(z - z_0)^m ,$$
$$L^k(z_0) \neq 0. \qquad (3.4)$$

To find coefficients $b_1, b_2, \ldots, b_n$ we will use the same technique as in the previous section. We substitute (3.4) into (3.3) and assume that $b_1 \neq 0$ and all the other coefficients are equal to zero, i.e. we will get the first approximation.

$$(z-z_0)=b_1\left[\begin{array}{l}\frac{L^k(z_0)}{k!}(z-z_0)^k+\frac{L^{k+1}(z_0)}{(k+1)!}(z-z_0)^{k+1}+\cdots+\\+\frac{L^m(z_0)}{m!}(z-z_0)^m\end{array}\right]^{\frac{1}{k}}, \tag{3.5}$$

$L^k(z_0)\neq 0.$

In the right-hand side we factor out $(z-z_0)$ from parentheses and neglect (in the first approximation) all the terms, except for $\frac{L^k}{k!}$.

As a result we have

$$(z-z_0)=b_1(z-z_0)\left[\begin{array}{l}\frac{L^k(z_0)}{k!}+\frac{L^{k+1}(z_0)}{(k+1)!}(z-z_0)+\\+\frac{L^{k+2}(z_0)}{(k+2)!}(z-z_0)^2+\ldots\end{array}\right]^{\frac{1}{k}}; \tag{3.6}$$

$$(z-z_0)=b_1(z-z_0)\sqrt[k]{\frac{L^k}{k!}},$$

and finally

$$b_1=\frac{1}{\sqrt[k]{\frac{L^k}{k!}}}. \tag{3.7}$$

The first approximation will be

$$z_1=z_0-\frac{1}{\sqrt[k]{\frac{L^k}{k!}}}\sqrt[k]{-L(z_0)}. \tag{3.8}$$

In finding the second approximation, we will suppose, that all the coefficients higher than b_2 are equal to zero. Then (3.3) takes on the form

$$z-z_0=\frac{1}{\sqrt[k]{\frac{L^k}{k!}}}\left[L(z)-L(z_0)\right]^{\frac{1}{k}}+b_2\left[L(z)-L(z_0)\right]^{\frac{2}{k}}. \tag{3.9}$$

We substitute (3.4) into (3.9) but suppose, that all the terms higher than $\frac{L^{k+1}}{(k+1)!}$ are equal to zero. In this case we get

$$z-z_0=\frac{1}{\sqrt[k]{\frac{L^k}{k!}}}\left[(z-z_0)^k\frac{L^k}{k!}+(z-z_0)^{k+1}\frac{L^{k+1}}{(k+1)!}+...\right]^{\frac{1}{k}}+$$

$$+b_2\left[(z-z_0)^k\frac{L^k}{k!}+(z-z_0)^{k+1}\frac{L^{k+1}}{(k+1)!}+...\right]^{\frac{2}{k}}. \tag{3.10}$$

In (3.10), we expand the expression inside the brackets into series of Newton binomial in terms of fractional powers

$$1=\left[1+\frac{1}{k}(z-z_0)\frac{L^{k+1}}{L^k}\frac{1}{k+1}+\cdots\right]+$$

$$+b_2(z-z_0)\left(\frac{L^k}{k!}\right)^{\frac{2}{k}}\left[1+\frac{2}{k}(z-z_0)\frac{L^{k+1}}{(k+1)!}\frac{k!}{L^k}+\cdots\right]. \tag{3.11}$$

Neglecting the higher terms of the expansion we make equal coefficients of $z-z_0$

$$0=\frac{1}{k}\frac{1}{k+1}\frac{L^{k+1}}{L^k}+b_2\left[\frac{L^k}{k!}\right]^{\frac{2}{k}}. \tag{3.12}$$

From (3.12) we find b_2

$$b_2=\frac{-\frac{L^{k+1}}{(k+1)!}}{k\left[\frac{L^k}{k!}\right]^{\frac{2}{k}+1}}. \tag{3.13}$$

The second approximation will be

$$z_2 = z_0 + \frac{1}{\sqrt[k]{\frac{L^k(z_0)}{k!}}}\sqrt[k]{-L(z_0)} + \frac{-\frac{L^{k+1}(z_0)}{(k+1)!}}{k\left[\frac{L^k(z_0)}{k!}\right]^{\frac{2}{k}+1}}\left[\sqrt[k]{-L(z_0)}\right]^2 \quad . \tag{3.14}$$

Similarly we find b_3 and the third approximation

$$b_3 = \frac{(3+k)\left[\frac{L^{k+1}}{(k+1)!}\right]^2 - 2\left[\frac{L^{k+2}}{(k+2)!}\right]\left[\frac{L^k}{(k-1)!}\right]}{2!k^2\left[\frac{L^k}{k!}\right]^{\frac{3}{k}+2}} \quad . \tag{3.15}$$

$$z_3 = z_0 + \frac{1}{\sqrt[k]{\frac{L^k(z_0)}{k!}}}\sqrt[k]{-L(z_0)} + \frac{\frac{-L^{k+1}(z_0)}{(k+1)!}}{k\left[\frac{L^k(z_0)}{k!}\right]^{(\frac{2}{k}+1)}}\left[\sqrt[k]{-L(z_0)}\right]^2 +$$

$$+\frac{(3+k)\left[\frac{L^{k+1}(z_0)}{(k+1)!}\right]^2 - 2\left[\frac{L^{k+2}(z_0)}{(k+2)!}\right]\left[\frac{L^k(z_0)}{(k-1)!}\right]}{2!k^2\left[\frac{L^k(z_0)}{k!}\right]^{\frac{3}{k}+2}}\left[\sqrt[k]{-L(z_0)}\right]^3 . \tag{3.16}$$

As in the two previous cases the coefficients of the series can be expressed in terms of nearly triangular determinants.

$$\Delta_2^{**} = \begin{vmatrix} \frac{L^{k+1}(z_0)}{(k+1)!} & \frac{2L^{k+2}(z_0)}{(k+2)!} \\ \frac{L^k(z_0)}{(k-1)!} & \frac{(3+k)L^{k+1}(z_0)}{(k+1)!} \end{vmatrix} . \tag{3.17}$$

Accordingly b_3 takes on the following form

$$b_3 = \frac{(3+k)\left[\frac{L^{k+1}}{(k+1)!}\right]^2 - 2\left[\frac{L^{k+2}}{(k+2)!}\right]\left[\frac{L^k}{(k-1)!}\right]}{2!k^2\left[\frac{L^k}{k!}\right]^{\frac{3}{k}+2}} = \frac{\Delta_2^{**}(z_0)}{2!k^2\left[\frac{L^k}{k!}\right]^{\frac{3}{k}+2}}. \tag{3.18}$$

The next coefficient of the series b_4 will be

$$b_4 = -\frac{\Delta_3^{**}(z_0)}{3!k^3\left[\frac{L^k}{k!}\right]^{\frac{4}{k}+3}}, \tag{3.19}$$

where $\Delta_3^{**}(z_0)$ is a nearly triangular determinant of the form

$$\Delta_3^{**} = \begin{vmatrix} \frac{L^{k+1}(z_0)}{(k+1)!} & \frac{2L^{k+2}(z_0)}{(k+2)!} & \frac{3L^{k+3}(z_0)}{(k+3)!} \\ \frac{L^k(z_0)}{(k-1)!} & \frac{(4+k)L^{k+1}(z_0)}{(k+1)!} & \frac{(8+k)L^{k+2}(z_0)}{(k+2)!} \\ 0 & \frac{2L^k(z_0)}{(k-1)!} & \frac{(4+2k)L^{k+1}(z_0)}{(k+1)!} \end{vmatrix}. \tag{3.20}$$

In the general form, coefficients $b_3, b_4, b_5, \ldots, b_n$, are expressed in terms of nearly triangular determinants as

$$b_n = (-1)^{n-1}\frac{\Delta_{n-1}^{**}(z_0)}{(n-1)!k^{n-1}\left[\frac{L^k(z_0)}{k!}\right]^{\frac{n+k(n-1)}{k}}}, \tag{3.21}$$

where $\Delta_{n-1}^{**}(z_0)$ are nearly triangular determinants of the form:

$$\Delta_n^{**} =$$

$$\begin{array}{cccccc}
\frac{1L^{(k+1)}}{(k+1)!} & \frac{2L^{(k+2)}}{(k+2)!} & \frac{3L^{(k+3)}}{(k+3)!} & \cdots & \frac{(k-2)L^{(n+k-2)}}{(n+k-2)!} & \frac{(n-1)L^{(n+k-1)}}{(n+k-1)!} \\
\frac{L^{(k)}}{(k-1)!} & \frac{(n+k)L^{(k+1)}}{(k+1)!} & \frac{(2n+k)L^{(k+2)}}{(k+2)!} & \cdots & \frac{((n-3)n+k)L^{(n+k-3)}}{(n+k-3)!} & \frac{((n-2)n+k)L^{(n+k-2)}}{(n+k-2)!} \\
0 & \frac{2L^{(k)}}{(k-1)!} & \frac{(n+2k)L^{(k+1)}}{(k+1)!} & \cdots & \frac{((n-4)n+2k)L^{(n+k-4)}}{(n+k-4)!} & \frac{((n-3)n+2k)L^{(n+k-3)}}{(n+k-3)!} \\
0 & 0 & \frac{2L^{(k)}}{(k-1)!} & \vdots & \vdots & \vdots \\
\cdots & \cdots & \cdots & \cdots & \frac{(n+(n-3)k)L^{(k+1)}}{(k+1)!} & \frac{(2n+(n-3)k)L^{(k+2)}}{(k+2)!} \\
0 & 0 & 0 & \vdots & \frac{(n-2)L^{(k)}}{(k-1)!} & \frac{(n+(n-2)k)L^{(k+1)}}{(k+1)!} \\
0 & 0 & 0 & \vdots & & \\
0 & 0 & 0 & \vdots & &
\end{array}$$

(3.22)

Finally, series (3.1) will be written as

$$z = z_0 + \frac{\sqrt[k]{-L(z_0)}}{\left[\frac{L^k(z_0)}{k!}\right]^{\frac{1}{k}}} +$$

$$+\sum_{n=2}^{\infty}(-1)^{n-1}\frac{\Delta_{n-1}^{**}(z_0)}{(n-1)!k^{n-1}\left[\frac{L^k(z_0)}{k!}\right]^{\frac{nk+n-k}{k}}}\left[\sqrt[k]{-L(z_0)}\right]^n,$$

$$L'(z_0) = L''(z_0) = \cdots = L^{k-1}(z_0) = 0, \quad L^k(z_0) \neq 0. \qquad (3.23)$$

Hence, we have managed to expand an algebraic function into infinite series for the vicinities of all types of points which an algebraic function can have. Using expansions (1.18), (2.14), and (3.23) we can get inverse function for any polynomial function in the vicinities of regular points, algebraic branch points and finite-degree poles.

4. Expression of the roots of a cubic equation in terms of its coefficients

Using a cubic equation as an example let us see how the method developed in the previous sections works. Consider the third-order polynomial function:

$$L(z) = z^3 + p_1 z^2 + p_2 z + p_3; \\ p_1, p_2, p_3 \in C. \tag{4.1}$$

We will use the method of expansion (1.18) and get expansion of the inverse function in the vicinity of the regular point *z=0*.

$$z = 0; \quad L(0) = p_3; \quad L'(0) = p_2; \quad L''(0) = 2p_1; \quad L'''(0) = 6. \\ p_1, p_2, p_3 \in C. \tag{4.2}$$

We find the values of the determinants (1.17) to be

$$\Delta_2 = \left[\frac{2p_1}{2!}\right] = p_1; \quad \Delta_3 = \begin{vmatrix} p_1 & 2 \\ p_2 & 4p_1 \end{vmatrix} = 4p_1^2 - 2p_2; \\ \Delta_4 = \begin{vmatrix} 2p_1 & 2 & 0 \\ p_2 & 5p_1 & 9 \\ 0 & 2p_2 & 6p_1 \end{vmatrix} = 60p_1^3 - 48p_1p_2; \quad \cdots \tag{4.3}$$

Then we substitute (4.3) into (1.18)

$$z = 0 - \frac{p_3}{p_2} - \frac{p_3^2}{1!\,p_2^3} p_1 - \frac{p_3^3}{2!\,p_2^5}\left[4p_1^2 - \frac{6p_2}{3}\right] - \\ - \frac{p_3^4}{3!\,p_2^7}\left[\frac{-48p_1p_2 + 60p_1^3}{4}\right] - \ldots . \tag{4.4}$$

The regularity of the coefficients of series (4.4) is expressed as

$$z = -\frac{p_3}{p_2}\left[\begin{array}{l} 1+\alpha_2+\alpha_3+\left[\frac{4}{2!}\alpha_2^2+\frac{5}{1!1!}\alpha_2\alpha_3+\frac{6}{2!}\alpha_3^2\right]+ \\ +\left[\frac{30}{3!}\alpha_2^3+\frac{42}{2!1!}\alpha_2^2\alpha_3+\frac{56}{2!1!}\alpha_2^2\alpha_3+\frac{72}{3!}\alpha_3^3\right]+\dots \end{array}\right], \tag{4.5}$$

$$\alpha_2 = \frac{p_1 p_3}{p_2^2}, \quad \alpha_3 = -\frac{p_3^2}{p_2^3}.$$

$$z = 1+\sum_{i=1}^{2}\alpha_{l_i} + \sum_{\substack{n=2;\\ r_1,r_2=0;\\ r_1+r_2=n}}^{\infty} \prod_{i=1}^{2}\frac{\alpha_{l_i}^{r_i}}{r_i!}\prod_{\nu=1}^{n-1}\left(\sum_{i=1}^{2} l_i r_i + 1-\nu\right),$$

$$l_1 = 2, \quad l_2 = 3.$$

Let us pass on to expanding the inverse function into series in the vicinity of the point at infinity which is a third order pole for the cubic polynomial. Accordingly, the expansion has the form of (2.14).

In accordance with (2.13)

$$\Delta_1 = p_1, \quad \Delta_2 = \begin{vmatrix} 2p_1 & 2p_2 \\ -3 & -2p_1 \end{vmatrix} = -2p_1^{\,2} + 6p_2,$$

$$\Delta_3 = \begin{vmatrix} p_1 & 2p_2 & 0 \\ -3 & -p_1 & 0 \\ 0 & -6 & -4p_1 \end{vmatrix} = 4p_1^3 - 18p_1p_2, \quad \cdots \tag{4.6}$$

We substitute (4.6) in (2.14)

$$z = 0 + \sqrt[3]{-p_3} - \frac{p_1}{1!3} - \frac{(-2p_1^2 + 6p_2)}{2!9}(\sqrt[3]{-p_3})^{-1} -$$

$$-\frac{4p_1^3 - 18p_1p_2}{3!27}(\sqrt[3]{-p_3})^{-2} - ....;$$

$$z = \sqrt[3]{-p_3}\begin{bmatrix} 1 + \frac{-p_1}{3\sqrt[3]{-p_3}} + \frac{-p_2}{3(\sqrt[3]{-p_3})^2} + \\ + \frac{p_1{}^2}{9(\sqrt[3]{-p_3})^2} - \frac{4p_1{}^3}{6(\sqrt[3]{-p_3})^3} + \\ + \frac{18p_1p_2}{6\cdot 27}(-p_3)^{-1} + ... \end{bmatrix}. \tag{4.7}$$

The regularity of the coefficients of series (4.7) is expressed as

$$z = \sqrt[3]{-p_3}\left[1 + \sum_{i=1}^{2}\beta_{ml_i} + \sum_{\substack{n=2 \\ r_1, r_2 = 0; \\ r_1 + r_2 = n.}}^{\infty} \prod_{i=1}^{2}\frac{\beta_{ml_i}^{r_i}}{r_i!}\prod_{\nu=1}^{n-1}\left(\sum_{i=1}^{2} l_i r_i + 1 - m\nu\right)\right];$$

$$m = 3; \quad l_1 = 2; \quad l_2 = 1; \tag{4.8}$$

$$\beta_{32} = \frac{-p_1}{3\sqrt[3]{-p_3}}; \quad \beta_{31} = \frac{-p_2}{3(\sqrt[3]{-p_3})^2}.$$

Convergence condition

$$abs(\beta_{32}) \le r_{32} \qquad abs(\beta_{31}) \le r_{31} \tag{4.9}$$

The domain of convergence can be found from the condition of having a common root of the cubic equation and its derivative

$$\begin{Bmatrix} L(z) = z^3 + p_1z^2 + p_2z + p_3 = 0 \\ L'(z) = z^2 + \frac{2}{3}p_1z + \frac{1}{3}p_2 = 0 \end{Bmatrix}. \tag{4.10}$$

The series (4.8) allows us to express all three roots of the cubic equation. This series converges in polycylinder (4.9). Outside the region (4.9), the roots of the cubic equation representing other series, that can be expressed similarly, using the expansion (1.18), (2.14), (3.23). Using this method, we can obtain an algebraic function, which is the solution of a cubic equation for all values of coefficients from field of complex numbers. This result is obtained, but here is not considered due to lack of space.
For other polynomials and more complicated algebraic equations, algebraic functions which are solutions of these equations can be obtained in the same way [12], [13].

Conclusion

This method of solving polynomials is a part of a more general research task concerned with finding solutions of nonlinear problems. Today the overwhelming majority of such problems are solved by numerical methods. Our aim is to develop analytical approaches to this task. Some interesting results have already been achieved. Thus, we have succeeded to get an analytical solution of the problem of vibrations of a plate moving in gas [15], to investigate analytically a characteristic solution of a complex electromechanical system [16]. A solution of the problem of a full analytical investigation of the dependence of polarization on the value of an external electric field in ferroelectrics has been obtained [14]. Another interesting line of investigations is analytical solution of systems of nonlinear algebraic equations. In this field some results have also been obtained. This research will be continued and in the short run one would expect some new interesting results.

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