A Note on the Degenerate High Order Daehee Polynomials

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Abstract

In this paper, we consider the degenerate high order Daehee polynomials which are derived from $p$-adic invariant integral on $\mathbb{Z}_p$ and investigate some properties of those polynomials.

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1. Introduction

Let $p$ be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of $p$-adic integers, the field of $p$-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The $p$-adic norm is normally defined by $|p|_p = \frac{1}{p}$. Let $f(x)$ be a uniformly differentiable function on $\mathbb{Z}_p$. Then the
\[ p\text{-adic invariant integral on } \mathbb{Z}_p \text{ is defined as} \]
\[ \int_{\mathbb{Z}_p} f(x) d\mu_0(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x) d\mu_0(x + p^N \mathbb{Z}_p) \]
\[ = \lim_{N \to \infty} \frac{1}{p^N} \sum_{n=0}^{p^N-1} f(x), \text{(see [12, 13, 15])}. \]

From (1.1), we have
\[ I_0(f_1) - I_0(f) = f'(0). \]  

As is well known, the Bernoulli polynomials are defined by the generating function to be
\[ \left( \frac{t}{e^t - 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}, \text{(see [1-2, 4-12])}. \]

When \( x = 0, B_n = B_n(0), (n \geq 0), \) are called the ordinary Bernoulli numbers.

In [2], L. Carlitz consider the degenerate Bernoulli polynomials which are given by the generating function to be
\[ \frac{t}{(1 + \lambda t)^x - 1} = \sum_{n=0}^{\infty} \beta_n(x|\lambda) \frac{t^n}{n!}. \]

When \( x = 0, \beta_n(\lambda) = \beta_n(0|\lambda) \) are called the degenerate Bernoulli numbers. Note that \( \lim_{\lambda \to 0} \beta_n(\lambda) = B_n. \)

The Daehee polynomials of order \( r \) are defined by
\[ D_n(x) = \int_{\mathbb{Z}_p} (x+y)_n d\mu_0(y), \text{ (n \geq 0), (see [6, 9, 10])}. \]

From (1.2) and (1.5), we can derive the generating function to be
\[ \sum_{n=0}^{\infty} D_n^{(r)}(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + t)^{x_1 + \cdots + x_r + x} d\mu_0(x_1) \cdots d\mu_0(x_r) \]
\[ = \left( \frac{\log(1 + t)}{t} \right)^r (1 + t)^x, \text{(see [6, 9])}. \]

(see [6, 9]).

By (1.3) and (1.6), it is not difficult to show that
\[ \left( \frac{\log(1 + t)}{t} \right)^r (1 + t)^{x-1} = \sum_{n=0}^{\infty} B_n^{(n+r+1)}(x) \frac{t^n}{n!}, \text{(see [6])}. \]

In this paper, we consider the degenerate high order Daehee numbers and polynomials which are derived from \( p\text{-adic invariant integral integral on } \mathbb{Z}_p \) and investigate some properties of those polynomials.
2. DEGENERATE DAEEHE POLYNOMIALS

Let us assume that $\lambda, t \in \mathbb{C}_p$ with $|\lambda t|^p < p^{-\frac{1}{r}}$. We define the *degenerate high order Daehee polynomials* by the generating function as follows:

$$
\sum_{n=0}^{\infty} D^{(k)}_{n,\lambda}(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(1 + \log(1 + \lambda t)^{\frac{1}{x}}\right)^{x_1 + \cdots + x_k + x} d\mu_0(x_1) \cdots d\mu_0(x_k)
= \left(\log \left(1 + \log(1 + \lambda t)^{\frac{1}{x}}\right)\right)^k \left(1 + \log(1 + \lambda t)^{\frac{1}{x}}\right)^x.
$$

(2.1)

When $x = 0, k = 1$, $D_{n,\lambda} = D_{n,\lambda}(0)$ are called the *n-th degenerate Daehee numbers*.

It is well-known fact that the generating function of the Stirling number of the first kind is given by

$$(\log(1 + t))^m = m! \sum_{l=m}^{\infty} S_1(l, m) \frac{t^l}{l!}, \text{ (see } [3, 8, 14]),$$

(2.2)

and the Stirling number of the second kind is defined by the generating function to be

$$(e^t - 1)^n = \sum_{m=n}^{\infty} S_2(m, n) \frac{t^m}{m!}, \text{ (see } [3, 14]).$$

By (2.1) and (2.2), we observe that

$$
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(1 + \log(1 + \lambda t)^{\frac{1}{x}}\right)^{x_1 + \cdots + x_k + x} d\mu_0(x_1) \cdots d\mu_0(x_k)
= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(x_1 + \cdots + x_k + x\right)^n d\mu_0(x_1) \cdots d\mu_0(x_k) \lambda^{-n} (\log(1 + \lambda t))^n
= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \lambda^{n-l} S_1(n, l) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k + x) \frac{d\mu_0(x_1) \cdots d\mu_0(x_k)}{n!}\right) \frac{t^n}{n!}.
$$

(2.3)

Thus, by (2.1) and (2.3), we obtain the following theorem.

**Theorem 2.1.** For $n \geq 0$, we have

$$D^{(k)}_{n,\lambda}(x) = \sum_{l=0}^{n} \lambda^{n-l} S_1(n, l) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k + x) \frac{d\mu_0(x_1) \cdots d\mu_0(x_k)}{n!}.$$
\[
\int_{Z_p} \cdots \int_{Z_p} (1 + t)^{x_1 + \cdots + x_k + x} d\mu_0(x_1) \cdots d\mu_0(x_k)
= \sum_{n=0}^{\infty} \left( \int_{Z_p} \cdots \int_{Z_p} (x_1 + \cdots + x_k + x)^n d\mu_0(x_1) \cdots d\mu_0(x_k) \right) \frac{t^n}{n!} \quad (2.4)
= \sum_{n=0}^{\infty} D_n^{(k)}(x) \frac{t^n}{n!}.
\]

Therefore, by Theorem 2.1 and (2.4), we obtain the following corollary.

**Corollary 2.2.** For \( n \geq 0 \), we have

\[
D_{n,\lambda}^{(k)}(x) = \sum_{l=0}^{n} \lambda^{n-l} S_1(n, l) D_l^{(k)}(x).
\]

By replacing \( t \) by \( \frac{1}{\lambda}(e^{\lambda t} - 1) \) in (2.1), we get

\[
\int_{Z_p} \cdots \int_{Z_p} (1 + t)^{x_1 + \cdots + x_k + x} d\mu_0(x_1) \cdots d\mu_0(x_k)
= \sum_{n=0}^{\infty} D_{n,\lambda}^{(k)}(x) \frac{1}{n!} \lambda^{-n} (e^{\lambda t} - 1)^n
= \sum_{n=0}^{\infty} D_{n,\lambda}^{(k)}(x) \frac{1}{n!} \lambda^{-n} \sum_{m=n}^{\infty} S_2(m, n) \frac{(\lambda t)^m}{m!}
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} D_{m,\lambda}^{(k)}(x) \frac{1}{n!} \lambda^{n-m} S_2(n, m) \right) \frac{t^n}{n!}. \quad (2.5)
\]

By (1.6) and (2.5), we obtain the following corollary.

**Corollary 2.3.** For \( n \geq 0 \), we have

\[
D_n^{(k)}(x) = \sum_{m=0}^{n} D_{m,\lambda}^{(k)}(x) \frac{1}{n!} \lambda^{n-m} S_2(n, m).
\]
By (1.7), we can derive the following equations easily:

\[
\left( \frac{\log \left( 1 + \log(1 + \lambda t)^{\frac{1}{x}} \right)}{\log(1 + \lambda t)^{\frac{1}{x}}} \right)^k \left( 1 + \log(1 + \lambda t)^{\frac{1}{x}} \right)^x
\]

\[
= \sum_{n=0}^{\infty} B_n^{(n+k+1)}(x+1) \frac{1}{n!} \left( \log(1 + \lambda t)^{\frac{1}{x}} \right)^n
\]

\[
= \sum_{n=0}^{\infty} B_n^{(n+k+1)}(x+1) \lambda^{-n} \sum_{l=n}^{\infty} S_1(l, n) \frac{(\lambda t)^l}{l!}
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} B_m^{(m+k+1)}(x+1) \lambda^{n-m} S_1(n, m) \right) \frac{t^n}{n!}.
\]

By (2.1) and (2.6), we obtain the following theorem.

**Theorem 2.4.** For \( n \geq 0 \), we have

\[
D_{n,\lambda}^{(k)}(x) = \sum_{m=0}^{n} B_m^{(m+k+1)}(x+1) \lambda^{n-m} S_1(n, m).
\]

We can observe that

\[
\left( \frac{\log \left( 1 + \log(1 + \lambda t)^{\frac{1}{x}} \right)}{\log(1 + \lambda t)^{\frac{1}{x}}} \right)^k \left( 1 + \log(1 + \lambda t)^{\frac{1}{x}} \right)^x
\]

\[
= \left( \sum_{n=0}^{\infty} D_{n,\lambda}^{(n+k+1)} \frac{t^n}{n!} \right) \left( \sum_{n=0}^{\infty} \binom{x}{n} \left( \frac{1}{\lambda} \log(1 + \lambda t) \right)^n \right)
\]

\[
= \left( \sum_{n=0}^{\infty} D_{n,\lambda}^{(n+k+1)} \frac{t^n}{n!} \right) \left( \sum_{m=0}^{\infty} \binom{m}{k} \sum_{k=0}^{m} \lambda^{-k}(x)_k S_1(m, k) \frac{(\lambda t)^m}{m!} \right)
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \sum_{l=0}^{m} D_{n-m,\lambda}^{(k)} \binom{x}{l} \binom{n}{m} \lambda^{m-l} S_1(m, l) \right) \frac{t^n}{n!}.
\]

Thus, by (2.1) and (2.7), we obtain the following theorem.

**Theorem 2.5.** For \( n \geq 0 \), we have

\[
D_{n,\lambda}^{(k)}(x) = \sum_{m=0}^{n} \sum_{l=0}^{m} D_{n-m,\lambda}^{(k)} \binom{x}{l} \binom{n}{m} \lambda^{m-l} S_1(m, l) D_{n-m,\lambda}^{(k)}.
\]
It is well-known that the high order Daehee polynomials of the second kind are defined by the generating function to be

\[
\int_{Z_p} \cdots \int_{Z_p} (1 + t)^{-\left(x_1 + \cdots + x_k + x\right)} d\mu_0(x_1) \cdots d\mu_0(x_k)
= \left( \frac{\log(1 + t)}{1 - (1 + t)^{-1}} \right)^k (1 + t)^x
= \sum_{n=0}^{\infty} \hat{D}_{n,k}(x) \frac{t^n}{n!},
\]

(see [6, 9, 10]).

From now on, we consider the degenerate high order Daehee polynomials of the second kind which are defined by the generating function to be

\[
\sum_{n=0}^{\infty} \hat{D}_{n,k}(x) \frac{t^n}{n!}
= \int_{Z_p} \cdots \int_{Z_p} \left( 1 + \log(1 + \lambda t)^{\frac{1}{\lambda}} \right)^{-x_1 - x_k + x} d\mu_0(x_1) \cdots d\mu_0(x_k)
= \left( \frac{\log \left( 1 + \log(1 + \lambda t)^{\frac{1}{\lambda}} \right)}{1 - \left( 1 + \log(1 + \lambda t)^{\frac{1}{\lambda}} \right)^{-1}} \right)^k \left( 1 + \log(1 + \lambda t)^{\frac{1}{\lambda}} \right)^x.
\]

(2.10)

By (2.10), we observe that

\[
\int_{Z_p} \cdots \int_{Z_p} \left( 1 + \log(1 + \lambda t)^{\frac{1}{\lambda}} \right)^{-x_1 - x_k + x} d\mu_0(x_1) \cdots d\mu_0(x_k)
= \sum_{n=0}^{\infty} \int_{Z_p} \cdots \int_{Z_p} \left( -x_1 - x_k + x \right)^{n} d\mu_0(x_1) \cdots d\mu_0(x_k)
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \lambda^{n-m} S_1(n, m) \int_{Z_p} \cdots \int_{Z_p} (-x_1 - x_k + x)^{m} d\mu_0(x_1) \cdots d\mu_0(x_k) \right) \frac{t^n}{n!}.
\]

(2.11)

By (2.10) and (2.11), we obtain the following theorem.

**Theorem 2.6.** For \( n \geq 0 \), we have

\[
\hat{D}_{n,k}(x) = \sum_{m=0}^{n} \lambda^{n-m} S_1(n, m) \hat{D}_{m}(x).
\]
By (2.10), we have
\[
\left( \frac{\log \left( 1 + \log(1 + \lambda t)^{\frac{1}{\lambda}} \right)}{\log(1 + \lambda t)^{\frac{1}{\lambda}}} \right)^k \left( 1 + \log(1 + \lambda t)^{\frac{1}{\lambda}} \right)^x
\]
\[
= \sum_{n=0}^{\infty} B_n^{(n+k+1)}(x+2) \left( \log(1 + \lambda t)^{\frac{1}{\lambda}} \right)^n \frac{n!}{n!}
\]
\[
= \sum_{n=0}^{\infty} B_n^{(n+k+1)}(x+2) \lambda^{-n} S_1(l, n) \left( \frac{\lambda t^l}{l!} \right)
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} B_m^{(m+k+1)}(x+2) \lambda^{n-m} S_1(n, m) \right) \frac{t^n}{n!}.
\]
(2.12)

Therefore, by (2.12), we obtain the following theorem.

**Theorem 2.7.** For \( n \geq 0 \), we have
\[
\hat{D}_{n, \lambda}^{(k)}(x) = \sum_{m=0}^{n} B_m^{(m+k+1)}(x+2) \lambda^{n-m} S_1(n, m).
\]

From (2.10) and (2.11), we have
\[
\hat{D}_{n, \lambda(k)}(x)
\]
\[
= \sum_{m=0}^{n} \lambda^{n-m} S_1(n, m) \int_{z_p} \cdots \int_{z_p} (-x_1 \cdots - x_k + x)_m d\mu_0(x_1) \cdots d\mu_0(x_k)
\]
\[
= \sum_{m=0}^{n} \lambda^{n-m} S_1(n, m) (-1)^m \int_{z_p} \cdots \int_{z_p} (x_1 \cdots + x_k - x + m - 1)_m d\mu_0(x_1) \cdots d\mu_0(x_k)
\]
\[
= \sum_{m=0}^{n} \lambda^{n-m} S_1(n, m) (-1)^m m! \int_{z_p} \cdots \int_{z_p} \binom{m-1}{m-l} (x_1 \cdots + x_k - x)_l d\mu_0(x_1) \cdots d\mu_0(x_k)
\]
\[
= \sum_{m=0}^{n} \lambda^{n-m} S_1(n, m) (-1)^m m! \int_{z_p} \cdots \int_{z_p} \sum_{l=0}^{m-l} (x_1 \cdots + x_k - x)_l d\mu_0(x_1) \cdots d\mu_0(x_k)
\]
\[
= \sum_{m=0}^{n} \lambda^{n-m} S_1(n, m) (-1)^m m! \int_{z_p} \cdots \int_{z_p} \sum_{l=0}^{m-l} (x_1 \cdots + x_k - x)_l D_l^{(k)}(-x)
\]
(2.13)

By (2.13), we obtain the following theorem.
Theorem 2.8. For \( n \geq 0 \), we have

\[
\hat{D}_{n,\lambda}^{(k)}(x) = \sum_{m=0}^{n} \sum_{l=1}^{m} \frac{(-1)^m}{l!} m! \lambda^{n-m} S_1(n, m) \binom{m-1}{m-l} D_l^{(k)}(-x).
\]

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