Applied Mathematical Sciences, Vol. 10, 2016, no. 52, 2589 - 2609
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ams.2016.66198

Strong Convergence of a Viscosity Iterative Algorithm in Banach Spaces with Applications

Suwicha Imnang[1]

Department of Mathematics and Statistics, Faculty of Science
Thaksin University, Phatthalung, 93210, Thailand
&
Centre of Excellence in Mathematics, CHE, Si Ayutthaya Road
Bangkok 10400, Thailand

Suthep Suantai

Department of Mathematics, Faculty of Science, Chiang Mai University
Chiang Mai, 50200, Thailand
&
Centre of Excellence in Mathematics, CHE, Si Ayutthaya Road
Bangkok 10400, Thailand

Abstract

We present the strong convergence theorems for the viscosity iterative scheme for finding a common element of the solution set of the system of general variational inequalities for two arbitrary nonlinear mappings and the fixed point set of a nonexpansive mapping in real 2-uniformly smooth and uniformly convex Banach spaces. Furthermore, we apply our main result with the problem of approximating a zero point of accretive operators and a fixed point of strictly pseudocontractive mappings in Banach spaces. The main results presented in this paper improve and extend some results in the literature.

Keywords: Strong convergence; General variational inequality; Zero point; Nonexpansive mapping

[1]Corresponding author

1 Introduction

Let X be a real Banach space and X^* be its dual space. Let C be a subset of X and let T be a self-mapping of C. We use $F(T)$ to denote the set of fixed points of T. Let $U = \{x \in X : \|x\| = 1\}$ be a unit sphere of X. X is said to be *uniformly convex* if for each $\epsilon \in (0, 2]$, there exists a constant $\delta > 0$ such that for any $x, y \in U$,

$$\|x - y\| \geq \epsilon \text{ implies } \|\frac{x+y}{2}\| \leq 1 - \delta.$$

The norm on X is said to be *Gâteaux differentiable* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.1}$$

exists for each $x, y \in U$ and in this case X is said to be *smooth*. X is said to have a *uniformly Frechet differentiable norm* if the limit (1.1) is attained uniformly for $x, y \in U$ and in this case X is said to be *uniformly smooth*. We define a function $\rho : [0, \infty) \to [0, \infty)$, called the *modulus of smoothness* of X, as follows:

$$\rho(\tau) = \sup\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x, y \in X, \|x\| = 1, \|y\| = \tau\}.$$

It is known that X is uniformly smooth if and only if $\lim_{\tau \to 0} \rho(\tau)/\tau = 0$. Let q be a fixed real number with $1 < q \leq 2$. Then a Banach space X is said to be *q-uniformly smooth* if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$. For $q > 1$, the generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \{f \in X^* : \langle x, f \rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\}, \quad \forall x \in X.$$

In particular, if $q = 2$, the mapping J_2 is called the *normalized duality mapping* (or *duality mapping*), and usually we write $J_2 = J$. If X is a Hilbert space, then $J = I$. Further, we have the following properties of the generalized duality mapping J_q:
(1) $J_q(x) = \|x\|^{q-2} J_2(x)$ for all $x \in X$ with $x \neq 0$.
(2) $J_q(tx) = t^{q-1} J_q(x)$ for all $x \in X$ and $t \in [0, \infty)$.
(3) $J_q(-x) = -J_q(x)$ for all $x \in X$.

It is known that if X is smooth, then J is a single-valued function, which is denoted by j. Recall that the duality mapping j is said to be *weakly sequentially continuous* if for each $\{x_n\} \subset X$ with $x_n \to x$ weakly, we have $j(x_n) \to j(x)$ weakly-$*$. We know that if X admits a weakly sequentially continuous duality mapping, then X is smooth. Recall that a mapping $f : C \to C$ is a *contraction* on C, if there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|, \quad \forall x, y \in C$. We use Π_C to denote the collection of all contractions on C. This is $\Pi_C = \{f | f : C \to C$ a contraction$\}$. A mapping $T : C \to C$ is said to be *nonexpansive*, if $\|T(x) - T(y)\| \leq \|x - y\|, \ \ \forall x, y \in C$. Let $A : C \to X$ be a nonlinear mapping. Then A is called

(i) *L-Lipschitz continuous* (or *Lipschitzian*) if there exists a constant $L \geq 0$ such that
$\|Ax - Ay\| \leq L\|x - y\|, \ \ \forall x, y \in C$;

(ii) *accretive* if there exists $j(x - y) \in J(x - y)$ such that
$\langle Ax - Ay, j(x - y)\rangle \geq 0, \ \ \forall x, y \in C$;

(iii) α- *inverse strongly accretive* if there exist $j(x - y) \in J(x - y)$ and $\alpha > 0$ such that
$\langle Ax - Ay, j(x - y)\rangle \geq \alpha\|Ax - Ay\|^2, \ \ \forall x, y \in C$;

(iv) *relaxed* (c, d)- *cocoercive* if there exist $j(x - y) \in J(x - y)$ and two constants $c, d \geq 0$ such that
$\langle Ax - Ay, j(x - y)\rangle \geq (-c)\|Ax - Ay\|^2 + d\|x - y\|^2, \ \ \forall x, y \in C$.

Let C be a nonempty closed convex subset of a real Hilbert space H. Recall that the classical variational inequality is to find $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \geq 0, \ \ \forall x \in C,$$

where $A : C \to H$ is a nonlinear mapping. Variational inequality theory has emerged as an important tool in studying a wide class of obstacle, unilateral, free, moving, equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework. The variational inequality problem has been extensively studied in the literature (see [1, 2, 3]).

In 2006, Aoyama et al. [4] first considered the following generalized variational inequality problem in Banach spaces. Let $A : C \to X$ be an accretive operator. Find a point $x^* \in C$ such that

$$\langle Ax^*, j(x - x^*)\rangle \geq 0, \ \ \forall x \in C. \tag{1.2}$$

The set of solutions of problem (1.2) denoted by $S(C, A)$. The problem (1.2) is very interesting as it is connected with the fixed point problem for nonlinear mapping and the problem of finding a zero point of an accretive operator in Banach spaces (see [4]). For the problem of finding a zero point of a nonlinear mapping (see [5, 6, 7]).

In 2010, Yao et al. [8] introduced the following system of general variational inequalities in Banach spaces. For given two operators $A_1, A_2 : C \to X$, they considered the problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle A_1y^* + x^* - y^*, j(x - x^*)\rangle \geq 0, \ \ \forall x \in C, \\ \langle A_2x^* + y^* - x^*, j(x - y^*)\rangle \geq 0, \ \ \forall x \in C, \end{cases} \tag{1.3}$$

which is called *the system of general variational inequalities in a real Banach space*. Recently, Katchang and Kumam [9] introduced the following system of general variational inequalities in Banach spaces. For given two operators $A_1, A_2 : C \to X$, they considered the problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda_1 A_1 y^* + x^* - y^*, j(x - x^*)\rangle \geq 0, & \forall x \in C, \\ \langle \lambda_2 A_2 x^* + y^* - x^*, j(x - y^*)\rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.4}$$

which is called *the system of general variational inequalities* in a real Banach space. The problem of finding solutions of (1.4) by using iterative methods has been studied by many others (see [10, 11, 12, 13]).

In this paper, motivated and inspired by the idea of Yao et al. [8] and Katchang and Kumam [9], we introduce a new iterative method for finding a common element of the set of solutions of the system of general variational inequalities in Banach spaces for two arbitrary nonlinear mappings and the set of fixed points of a nonexpansive mapping in real 2- uniformly smooth and uniformly convex Banach spaces. We prove the strong convergence of the proposed iterative algorithm without the condition of weakly sequentially continuous duality mapping. Our result improves and extends the recent results of Yao et al. [8] and Katchang and Kumam [9].

2 Preliminaries

In this section, we recall the well-known results and give some useful lemmas that are used in the next section.

Lemma 2.1. (see [14]). *Let X be a q-uniformly smooth Banach space with $1 \leq q \leq 2$. Then*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + 2\|Ky\|^q$$

for all $x, y \in X$, where K is the q-uniformly smooth constant of X.

Lemma 2.2. (see [15]). *In a Banach space X, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall x, y \in X,$$

where $j(x + y) \in J(x + y)$.

Lemma 2.3. (see [16]). Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that

$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad n \geq 1,$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that
(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Let C be a nonempty closed convex subset of a smooth Banach space X and let D be a nonempty subset of C. A mapping $Q : C \to D$ is said to be *sunny* if

$$Q\big(Qx + t(x - Qx)\big) = Qx,$$

whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. A mapping $Q : C \to D$ is called a *retraction* if $Qx = x$ for all $x \in D$. Furthermore, Q is a *sunny nonexpansive retraction* from C onto D if Q is a retraction from C onto D, which is also sunny and nonexpansive. A subset D of C is called a sunny nonexpansive retraction of C if there exists a sunny nonexpansive retraction from C onto D.

It is well known that if X is a Hilbert space, then a sunny nonexpansive retraction Q_C is coincident with the metric projection from X onto C.

Lemma 2.4. (see [17]). Let C be a closed convex subset of a smooth Banach space X. Let D be a nonempty subset of C and $Q : C \to D$ be a retraction. Then the following are equivalent:
(a) Q is sunny and nonexpansive.
(b) $\|Qx - Qy\|^2 \leq \langle x - y, j(Qx - Qy)\rangle \ \forall x, y \in C$.
(c) $\langle x - Qx, j(y - Qx)\rangle \leq 0$, $\forall x \in C, y \in D$.

Lemma 2.5. (see [18]). If X is strictly convex and uniformly smooth and if $T : C \to C$ is a nonexpansive mapping having a nonempty fixed point set $F(T)$, then the set $F(T)$ is a sunny nonexpansive retraction of C.

Lemma 2.6. (see [19]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{b_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$. Suppose that $x_{n+1} = (1-b_n)y_n + b_n x_n$ for all integers $n \geq 1$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.7. (see [20]). *Let C be a closed convex subset of a strictly convex Banach space X. Let T_1 and T_2 be two nonexpansive mappings from C into itself with $F(T_1) \bigcap F(T_2) \neq \emptyset$. Define a mapping S by*

$$Sx = \lambda T_1 x + (1 - \lambda) T_2 x, \quad \forall x \in C,$$

where λ is a constant in $(0,1)$. Then S is nonexpansive and $F(S) = F(T_1) \bigcap F(T_2)$.

Lemma 2.8. (see [21]). *Let X be a real smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and $g(\|x - y\|) \leq \|x\|^2 - 2\langle x, j(y)\rangle + \|y\|^2$ for all $x, y \in B_r$.*

Lemma 2.9. (see [16]). *Let X be a uniformly smooth Banach space, C be a closed convex subset of X, $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and let $f \in \Pi_C$. Then the sequence $\{x_t\}$ defined by $x_t = tf(x_t) + (1-t)Tx_t$ converges strongly to a point in F(T) as $t \to 0$. If we define a mapping $Q : \Pi_C \to F(T)$ by $Q(f) := \lim_{t\to 0} x_t, \quad \forall f \in \Pi_C$, then $Q(f)$ solves the following variational inequality:*

$$\langle (I-f)Q(f), j(Q(f)-p)\rangle \leq 0, \quad \forall f \in \Pi_C,\ p \in F(T).$$

Next, we prove a lemma which is very useful for our consideration.

Lemma 2.10. *Let C be a nonempty closed convex subset of a real Banach space X and let $\lambda_1, \lambda_2 > 0$ and $A_1, A_2 : C \to X$ be two mappings. Let $G : C \to C$ be defined by*

$$G(x) = Q_C[Q_C(x - \lambda_2 A_2 x) - \lambda_1 A_1 Q_C(x - \lambda_2 A_2 x)], \quad \forall x \in C.$$

If $I - \lambda_1 A_1$ and $I - \lambda_2 A_2$ are nonexpansive mappings, then G is nonexpansive.

Proof. For any $x, y \in C$, we have

$$\begin{aligned}
\|G(x) - G(y)\| &= \|Q_C[Q_C(x - \lambda_2 A_2 x) - \lambda_1 A_1 Q_C(x - \lambda_2 A_2 x)] \\
&\qquad - Q_C[Q_C(y - \lambda_2 A_2 y) - \lambda_1 A_1 Q_C(y - \lambda_2 A_2 y)]\| \\
&\leq \|Q_C(x - \lambda_2 A_2 x) - \lambda_1 A_1 Q_C(x - \lambda_2 A_2 x) \\
&\qquad - (Q_C(y - \lambda_2 A_2 y) - \lambda_1 A_1 Q_C(y - \lambda_2 A_2 y)\| \\
&= \|(I - \lambda A_1) Q_C (I - \lambda_2 A_2) x - (I - \lambda_1 A_1) Q_C (I - \lambda_2 A_2) y\| \\
&\leq \|x - y\|.
\end{aligned}$$

This show that G is a nonexpansive mapping. □

Lemma 2.11. (see [9]). *Let C be a nonempty closed convex subset of a real smooth Banach space X. Let Q_C be the sunny nonexpansive retraction from X onto C. Let $A_1, A_2 : C \to X$ be two possibly nonlinear mappings. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem (1.4) if and only if $x^* = Q_C(y^* - \lambda_1 A_1 y^*)$ where $y^* = Q_C(x^* - \lambda_2 A_2 x^*)$.*

Remark 2.1. From Lemma 2.11, we note that

$$x^* = Q_C[Q_C(x^* - \lambda_2 A_2 x^*) - \lambda_1 A_1 Q_C(x^* - \lambda_2 A_2 x^*)],$$

which implies that x^* is a fixed point of the mapping G, which defined as in Lemma 2.10.

3 Main results

We are now in a position to state and prove our main result.

Theorem 3.1. *Let X be a uniformly convex and 2-uniformly smooth Banach space with the 2-uniformly smooth constant K, let C be a nonempty closed convex subset of X and Q_C be a sunny nonexpansive retraction from X onto C. Let $A_1, A_2 : C \to X$ be two mappings. Let f be a contractive mapping with the constant $\alpha \in (0,1)$ and let $S : C \to C$ be a nonexpansive mapping such that $\Omega = F(S) \bigcap F(G) \neq \emptyset$, where G is the mapping defined as in Lemma 2.10. For a given $x_1 \in C$, let $\{x_n\}$ and $\{y_n\}$ be the sequences generated by*

$$\begin{cases} y_n = Q_C(x_n - \lambda_2 A_2 x_n), \\ x_{n+1} = a_n f(x_n) + b_n x_n + c_n S Q_C(y_n - \lambda_1 A_1 y_n), \quad n \geq 1, \end{cases}$$

where $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are three sequences in $(0,1)$ such that
(i) $a_n + b_n + c_n = 1, \quad \forall n \geq 1$;
(ii) $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=1}^{\infty} a_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$.
If $I - \lambda_1 A_1$, $I - \lambda_2 A_2$ are nonexpansive and $\lim_{n\to\infty} \|A_1 y_n - A_1 y^*\| = \lim_{n\to\infty} \|A_2 x_n - A_2 x^*\| = 0$ for all $x^* \in \Omega$ and $y^* = Q_C(x^* - \lambda_2 A_2 x^*)$. Then $\{x_n\}$ converges strongly to $q \in \Omega$, which solves the following variational inequality:

$$\langle q - f(q), j(q-p) \rangle \leq 0, \quad \forall f \in \Pi_C, \ p \in \Omega.$$

Proof. Step 1. We show that $\{x_n\}$ is bounded.
Let $x^* \in \Omega$ and $t_n = Q_C(y_n - \lambda_1 A_1 y_n)$. It follows from Lemma 2.11 that

$$x^* = Q_C[Q_C(x^* - \lambda_2 A_2 x^*) - \lambda_1 A_1 Q_C(x^* - \lambda_2 A_2 x^*)].$$

Put $y^* = Q_C(x^* - \lambda_2 A_2 x^*)$, then $x^* = Q_C(y^* - \lambda_1 A_1 y^*)$ and

$$x_{n+1} = a_n f(x_n) + b_n x_n + c_n S t_n.$$

Since $I - \lambda_i A_i$ $(i = 1, 2)$ and Q_C are nonexpansive. Therefore

$$\begin{aligned} \|t_n - x^*\| &= \|Q_C(y_n - \lambda_1 A_1 y_n) - Q_C(y^* - \lambda_1 A_1 y^*)\| \leq \|y_n - y^*\| \\ &= \|Q_C(x_n - \lambda_2 A_2 x_n) - Q_C(x^* - \lambda_2 A_2 x^*)\| \leq \|x_n - x^*\| \end{aligned} \tag{3.1}$$

and $\|St_n - x^*\| \leq \|t_n - x^*\|$. It follows that

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|a_n f(x_n) + b_n x_n + c_n S t_n - x^*\| \\ &\leq a_n \|f(x_n) - x^*\| + b_n \|x_n - x^*\| + c_n \|t_n - x^*\| \\ &\leq a_n \|f(x_n) - x^*\| + (1 - a_n) \|x_n - x^*\| \\ &\leq a_n \alpha \|x_n - x^*\| + a_n \|f(x^*) - x^*\| + (1 - a_n) \|x_n - x^*\| \\ &= a_n \|f(x^*) - x^*\| + (1 - a_n(1 - \alpha)) \|x_n - x^*\|. \end{aligned}$$

By induction, we have

$$\|x_{n+1} - x^*\| \leq \max\{\frac{\|f(x^*) - x^*\|}{1-\alpha}, \|x_1 - x^*\|\}.$$

Therefore, $\{x_n\}$ is bounded. Hence $\{y_n\}$, $\{t_n\}$, $\{A_1 y_n\}$, $\{A_2 x_n\}$, $\{St_n\}$ and $\{f(x_n)\}$ are also bounded.

Step 2. We show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.
By nonexpansiveness of Q_C and $I - \lambda_i A_i$ $(i = 1, 2)$, we have

$$\begin{aligned} \|t_{n+1} - t_n\| &= \|Q_C(y_{n+1} - \lambda_1 A_1 y_{n+1}) - Q_C(y_n - \lambda_1 A_1 y_n)\| \\ &\leq \|y_{n+1} - y_n\| = \|Q_C(x_{n+1} - \lambda_2 A_2 x_{n+1}) - Q_C(x_n - \lambda_2 A_2 x_n)\| \\ &\leq \|x_{n+1} - x_n\|. \end{aligned} \tag{3.2}$$

Let $w_n = \frac{x_{n+1} - b_n x_n}{1 - b_n}$, $n \in \mathbb{N}$. Then $x_{n+1} = b_n x_n + (1 - b_n) w_n$ for all $n \in \mathbb{N}$ and

$$\begin{aligned} w_{n+1} - w_n &= \frac{x_{n+2} - b_{n+1} x_{n+1}}{1 - b_{n+1}} - \frac{x_{n+1} - b_n x_n}{1 - b_n} \\ &= \frac{a_{n+1} f(x_{n+1}) + c_{n+1} St_{n+1}}{1 - b_{n+1}} - \frac{a_n f(x_n) + c_n St_n}{1 - b_n} \\ &= \frac{a_{n+1}}{1 - b_{n+1}}(f(x_{n+1}) - St_{n+1}) + \frac{a_n}{1 - b_n}(St_n - f(x_n)) + St_{n+1} - St_n. \end{aligned} \tag{3.3}$$

By (3.2), (3.3) and nonexpansiveness of S, we have

$$\|w_{n+1} - w_n\| - \|x_{n+1} - x_n\| \leq \frac{a_{n+1}}{1 - b_{n+1}}\|f(x_{n+1}) - St_{n+1}\| + \frac{a_n}{1 - b_n}\|St_n - f(x_n)\|.$$

By this together with the conditions (ii) and (iii), we obtain that

$$\limsup_{n\to\infty} \|w_{n+1} - w_n\| - \|x_{n+1} - x_n\| \leq 0.$$

Hence, by Lemma 2.6, we get $\|x_n - w_n\| \to 0$ as $n \to \infty$. Consequently,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1 - b_n)\|w_n - x_n\| = 0. \tag{3.4}$$

Step 3. We show that $\lim_{n\to\infty} \|Sx_n - x_n\| = 0$.
Since

$$x_{n+1} - x_n = a_n(f(x_n) - x_n) + c_n(St_n - x_n),$$

it follows from (3.4) and the conditions (i)-(iii) that

$$\|St_n - x_n\| \to 0$$

as $n \to \infty$.(3.5)Let $r = \sup_{n\geq 1}\{\|x_n - x^*\|, \|y_n - y^*\|, \|t_n - x^*\|\}$. By Lemma 2.4 (b) and Lemma 2.8, we obtain

$$\begin{aligned}\|t_n - x^*\|^2 &= \|Q_C(y_n - \lambda_1 A_1 y_n) - Q_C(y^* - \lambda_1 A_1 y^*)\|^2 \\ &\leq \langle y_n - \lambda_1 A_1 y_n - (y^* - \lambda_1 A_1 y^*), j(t_n - x^*)\rangle \\ &= \langle y_n - y^*, j(t_n - x^*)\rangle - \lambda_1 \langle A_1 y_n - A_1 y^*, j(t_n - x^*)\rangle \\ &\leq \frac{1}{2}[\|y_n - y^*\|^2 + \|t_n - x^*\|^2 - g(\|y_n - y^* - (t_n - x^*)\|)] \\ &\quad + \lambda_1 \langle A_1 y^* - A_1 y_n, j(t_n - x^*)\rangle,\end{aligned}$$

which implies

$$\begin{aligned}\|t_n - x^*\|^2 &\leq \|y_n - y^*\|^2 - g(\|y_n - y^* - (t_n - x^*)\|) \\ &\quad + 2\lambda_1 \langle A_1 y^* - A_1 y_n, j(t_n - x^*)\rangle \\ &\leq \|y_n - y^*\|^2 - g(\|y_n - y^* - (t_n - x^*)\|) \\ &\quad + 2\lambda_1 \|A_1 y^* - A_1 y_n\|\|t_n - x^*\|. \end{aligned} \tag{3.6}$$

Similarly, we have

$$\begin{aligned}\|y_n - y^*\|^2 &= \|Q_C(x_n - \lambda_2 A_2 x_n) - Q_C(x^* - \lambda_2 A_2 x^*)\|^2 \\ &\leq \langle x_n - \lambda_2 A_2 x_n - (x^* - \lambda_2 A_2 x^*), j(y_n - y^*)\rangle \\ &= \langle x_n - x^*, j(y_n - y^*)\rangle - \lambda_2 \langle A_2 x_n - A_2 x^*, j(y_n - y^*)\rangle \\ &\leq \frac{1}{2}[\|x_n - x^*\|^2 + \|y_n - y^*\|^2 - g(\|x_n - x^* - (y_n - y^*)\|)] \\ &\quad + \lambda_2 \langle A_2 x^* - A_2 x_n, j(y_n - y^*)\rangle,\end{aligned}$$

which implies

$$\begin{aligned}\|y_n - y^*\|^2 &\leq \|x_n - x^*\|^2 - g(\|x_n - x^* - (y_n - y^*)\|) \\ &\quad + 2\lambda_2 \langle A_2 x^* - A_2 x_n, j(y_n - y^*)\rangle \\ &\leq \|x_n - x^*\|^2 - g(\|x_n - x^* - (y_n - y^*)\|) \\ &\quad + 2\lambda_2 \|A_2 x^* - A_2 x_n\|\|y_n - y^*\|. \end{aligned} \tag{3.7}$$

From (3.6), (3.7) and the convexity of $\|.\|^2$, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq a_n\|f(x_n) - x^*\|^2 + b_n\|x_n - x^*\|^2 + c_n\|t_n - x^*\|^2 \\
&\leq a_n\|f(x_n) - x^*\|^2 + b_n\|x_n - x^*\|^2 \\
&\quad + c_n[\|y_n - y^*\|^2 - g(\|y_n - y^* - (t_n - x^*)\|) \\
&\quad + 2\lambda_1\|A_1y^* - A_1y_n\|\|t_n - x^*\|] \\
&\leq a_n\|f(x_n) - x^*\|^2 + b_n\|x_n - x^*\|^2 \\
&\quad + c_n[\|x_n - x^*\|^2 - g(\|x_n - x^* - (y_n - y^*)\|) \\
&\quad + 2\lambda_2\|A_2x^* - A_2x_n\|\|y_n - y^*\| - g(\|y_n - y^* - (t_n - x^*)\|) \\
&\quad + 2\lambda_1\|A_1y^* - A_1y_n\|\|t_n - x^*\|] \\
&= a_n\|f(x_n) - x^*\|^2 + (1 - a_n)\|x_n - x^*\|^2 \\
&\quad + 2c_n\lambda_1\|A_1y^* - A_1y_n\|\|t_n - x^*\| + 2c_n\lambda_2\|A_2x^* - A_2x_n\|\|y_n - y^*\| \\
&\quad - c_ng(\|y_n - y^* - (t_n - x^*)\|) - c_ng(\|x_n - x^* - (y_n - y^*)\|),
\end{aligned}$$

which implies

$$\begin{aligned}
&c_ng(\|y_n - y^* - (t_n - x^*)\|) + c_ng(\|x_n - x^* - (y_n - y^*)\|) \\
&\leq a_n\|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\quad + 2c_n\lambda_1\|A_1y^* - A_1y_n\|\|t_n - x^*\| + 2c_n\lambda_2\|A_2x^* - A_2x_n\|\|y_n - y^*\| \\
&\leq a_n\|f(x_n) - x^*\|^2 + \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) \\
&\quad + 2c_n\lambda_1\|A_1y^* - A_1y_n\|\|t_n - x^*\| + 2c_n\lambda_2\|A_2x^* - A_2x_n\|\|y_n - y^*\|.
\end{aligned}$$

By the conditions (ii)-(iii), (3.4) and $\lim_{n\to\infty}\|A_1y_n - A_1y^*\| = \lim_{n\to\infty}\|A_2x_n - A_2x^*\| = 0$, we obtain

$$\lim_{n\to\infty} g(\|y_n - y^* - (t_n - x^*)\|) = 0 \text{ and } \lim_{n\to\infty} g(\|x_n - x^* - (y_n - y^*)\|) = 0.$$

It follows from the properties of g that

$$\lim_{n\to\infty} \|y_n - y^* - (t_n - x^*)\| = 0 \text{ and } \lim_{n\to\infty} \|x_n - x^* - (y_n - y^*)\| = 0.$$

Hence

$$\|x_n - t_n\| \leq \|x_n - y_n - (x^* - y^*)\| + \|y_n - t_n - (y^* - x^*)\| \to 0 \quad \text{as } n \to \infty. \tag{3.8}$$

By (3.5) and (3.8), we have

$$\begin{aligned}
\|Sx_n - x_n\| &\leq \|Sx_n - St_n\| + \|St_n - x_n\| \\
&\leq \|x_n - t_n\| + \|St_n - x_n\| \to 0 \quad \text{as } n \to \infty. \qquad (3.9)
\end{aligned}$$

Define a mapping $W : C \to C$ as

$$Wx = \eta Sx + (1-\eta)Gx, \quad \forall x \in C,$$

where η is a constant in $(0,1)$. Then, it follows from Lemma 2.7 that $F(W) = F(G)\bigcap F(S)$ and W is nonexpansive. From (3.8) and (3.9), we have

$$\begin{aligned} \|x_n - Wx_n\| &= \|x_n - (\eta Sx_n + (1-\eta)Gx_n)\| \\ &= \|\eta(x_n - Sx_n) + (1-\eta)(x_n - Gx_n)\| \\ &\le \eta\|x_n - Sx_n\| + (1-\eta)\|x_n - Gx_n\| \\ &= \eta\|x_n - Sx_n\| + (1-\eta)\|x_n - t_n\| \to 0 \ \text{ as } n \to \infty. \end{aligned} \tag{3.10}$$

Step 4. We claim that

$$\limsup_{n\to\infty}\langle f(q) - q, j(x_n - q)\rangle \le 0, \tag{3.11}$$

where $q = \lim_{t\to 0} x_t$ with x_t being the fixed point of the contraction

$$x \mapsto tf(x) + (1-t)Wx.$$

From Lemma 2.9, we have $q \in F(W) = F(G)\bigcap F(S) = \Omega$ and

$$\langle (I-f)q, j(q-p)\rangle \le 0, \quad \forall f \in \Pi_C, \ p \in \Omega.$$

Since $x_t = tf(x_t) + (1-t)Wx_t$, we have

$$\begin{aligned} \|x_t - x_n\| &= \|tf(x_t) + (1-t)Wx_t - x_n\| \\ &= \|(1-t)(Wx_t - x_n) + t(f(x_t) - x_n)\|. \end{aligned}$$

It follows from (3.10) and Lemma 2.2 that

$$\begin{aligned} &\|x_t - x_n\|^2 = \|(1-t)(Wx_t - x_n) + t(f(x_t) - x_n)\|^2 \\ &\le (1-t)^2\|Wx_t - x_n\|^2 + 2t\langle f(x_t) - x_n, j(x_t - x_n)\rangle \\ &\le (1-t)^2(\|Wx_t - Wx_n\| + \|Wx_n - x_n\|)^2 + 2t\langle f(x_t) - x_n, j(x_t - x_n)\rangle \\ &= (1-t)^2(\|Wx_t - Wx_n\|^2 + 2\|Wx_t - Wx_n\|\|Wx_n - x_n\| + \|Wx_n - x_n\|^2) \\ &\quad + 2t\langle f(x_t) - x_t, j(x_t - x_n)\rangle + 2t\langle x_t - x_n, j(x_t - x_n)\rangle \\ &\le (1-2t+t^2)\|x_t - x_n\|^2 + (1-t)^2(2\|x_t - x_n\|\|Wx_n - x_n\| + \|Wx_n - x_n\|^2) \\ &\quad + 2t\langle f(x_t) - x_t, j(x_t - x_n)\rangle + 2t\|x_t - x_n\|^2 \\ &= (1+t^2)\|x_t - x_n\|^2 + f_n(t) + 2t\langle f(x_t) - x_t, j(x_t - x_n)\rangle, \end{aligned} \tag{3.12}$$

where $f_n(t) = (1-t)^2(2\|x_t - x_n\| + \|Wx_n - x_n\|)\|Wx_n - x_n\| \to 0$ as $n \to \infty$. It follows from (3.12) that

$$\langle x_t - f(x_t), j(x_t - x_n)\rangle \le \frac{t}{2}\|x_t - x_n\|^2 + \frac{f_n(t)}{2t}. \tag{3.13}$$

Let $n \to \infty$ in (3.13), we obtain that

$$\limsup_{n\to\infty}\langle x_t - f(x_t), j(x_t - x_n)\rangle \le \frac{t}{2}M, \tag{3.14}$$

where $M > 0$ is a constant such that $M \ge \|x_t - x_n\|^2$ for all $t \in (0,1)$ and $n \ge 1$. Let $t \to 0$ in (3.14), we obtain

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle x_t - f(x_t), j(x_t - x_n)\rangle \le 0. \tag{3.15}$$

On the other hand, we have

$$\begin{aligned}\langle f(q) - q, j(x_n - q)\rangle &= \langle f(q) - q, j(x_n - q)\rangle - \langle f(q) - q, j(x_n - x_t)\rangle \\ &\quad + \langle f(q) - q, j(x_n - x_t)\rangle - \langle f(q) - x_t, j(x_n - x_t)\rangle \\ &\quad + \langle f(q) - x_t, j(x_n - x_t)\rangle - \langle f(x_t) - x_t, j(x_n - x_t)\rangle \\ &\quad + \langle f(x_t) - x_t, j(x_n - x_t)\rangle \\ &= \langle f(q) - q, j(x_n - q) - j(x_n - x_t)\rangle + \langle x_t - q, j(x_n - x_t)\rangle \\ &\quad + \langle f(q) - f(x_t), j(x_n - x_t)\rangle + \langle f(x_t) - x_t, j(x_n - x_t)\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}&\limsup_{n\to\infty}\langle f(q) - q, j(x_n - q)\rangle \\ &\le \limsup_{n\to\infty}\langle f(q) - q, j(x_n - q) - j(x_n - x_t)\rangle + \|x_t - q\|\limsup_{n\to\infty}\|x_n - x_t\| \\ &\quad + \alpha\|x_t - q\|\limsup_{n\to\infty}\|x_n - x_t\| + \limsup_{n\to\infty}\langle f(x_t) - x_t, j(x_n - x_t)\rangle.\end{aligned}$$

Noticing that j is norm-to-norm uniformly continuous on a bounded subset of C, it follows from (3.15) and $\lim_{t\to 0} x_t = q$ that

$$\limsup_{n\to\infty}\langle f(q) - q, j(x_n - q)\rangle = \limsup_{t\to 0}\limsup_{n\to\infty}\langle f(q) - q, j(x_n - q)\rangle \le 0.$$

Hence (3.11) holds.

Step 5. Finally, we show that $x_n \to q$ as $n \to \infty$.
From (3.1), we have

$$\begin{aligned}\|x_{n+1} - q\|^2 &= \langle x_{n+1} - q, j(x_{n+1} - q)\rangle \\ &= \langle a_n(f(x_n) - q) + b_n(x_n - q) + c_n(St_n - q), j(x_{n+1} - q)\rangle \\ &= a_n\langle f(x_n) - f(q), j(x_{n+1} - q)\rangle + b_n\langle x_n - q, j(x_{n+1} - q)\rangle \\ &\quad + c_n\langle St_n - q, j(x_{n+1} - q)\rangle + a_n\langle f(q) - q, j(x_{n+1} - q)\rangle\end{aligned}$$

$$
\begin{aligned}
&\le a_n\alpha\|x_n-q\|\|x_{n+1}-q\|+b_n\|x_n-q\|\|x_{n+1}-q\| \\
&\quad +c_n\|St_n-q\|\|x_{n+1}-q\|+a_n\langle f(q)-q, j(x_{n+1}-q)\rangle \\
&\le a_n\alpha\|x_n-q\|\|x_{n+1}-q\|+b_n\|x_n-q\|\|x_{n+1}-q\| \\
&\quad +c_n\|x_n-q\|\|x_{n+1}-q\|+a_n\langle f(q)-q, j(x_{n+1}-q)\rangle \\
&=(1-a_n(1-\alpha))\|x_n-q\|\|x_{n+1}-q\|+a_n\langle f(q)-q, j(x_{n+1}-q)\rangle \\
&\le \frac{1-a_n(1-\alpha)}{2}(\|x_n-q\|^2+\|x_{n+1}-q\|^2)+a_n\langle f(q)-q, j(x_{n+1}-q)\rangle \\
&\le \frac{1-a_n(1-\alpha)}{2}\|x_n-q\|^2+\frac{1}{2}\|x_{n+1}-q\|^2+a_n\langle f(q)-q, j(x_{n+1}-q)\rangle,
\end{aligned}
$$

which implies

$$\|x_{n+1}-q\|^2\le(1-a_n(1-\alpha))\|x_n-q\|^2+a_n(1-\alpha)\frac{2\langle f(q)-q, j(x_{n+1}-q)\rangle}{1-\alpha}.$$

It follows from Lemma 2.3, (3.11) and the condition (ii) that $\{x_n\}$ converges strongly to q. This completes the proof. □

The following examples show that there are mappings A_1 and A_2 which satisfy those conditions in Theorem 3.1.

Let X be a uniformly convex and 2-uniformly smooth Banach space with the 2-uniformly smooth constant K, let C be a nonempty closed convex subset of X. Let $A_1, A_2 : C \to X$ be α_1-inverse strongly accretive and α_2-inverse strongly accretive, respectively. If $0<\lambda_1<\frac{\alpha_1}{K^2}$ and $0<\lambda_2<\frac{\alpha_2}{K^2}$, then we have

(1) $I-\lambda_1A_1$ and $I-\lambda_2A_2$ are nonexpansive and

(2) $\|A_1y_n-A_1y^*\|\to 0$ and $\|A_2x_n-A_2x^*\|\to 0$ as $n\to\infty$ for all $x^*\in\Omega$ and $y^*=Q_C(x^*-\lambda_2A_2x^*)$ where $\{x_n\}$ and $\{y_n\}$ are two sequences defined as in Theorem 3.1.

Proof. (1) For any $x, y\in C$, it follows from Lemma 2.1 that

$$
\begin{aligned}
&\|(I-\lambda_1A_1)x-(I-\lambda_1A_1)y\|^2=\|x-y-\lambda_1(A_1x-A_1y)\|^2 \\
&\le \|x-y\|^2-2\lambda_1\langle A_1x-A_1y, j(x-y)\rangle+2{\lambda_1}^2K^2\|A_1x-A_1y\|^2 \\
&\le \|x-y\|^2-2\lambda_1\alpha_1\|A_1x-A_1y\|^2+2{\lambda_1}^2K^2\|A_1x-A_1y\|^2 \\
&=\|x-y\|^2+2\lambda_1(\lambda_1K^2-\alpha_1)\|A_1x-A_1y\|^2.
\end{aligned}
$$

It clear that if $0<\lambda_1<\frac{\alpha_1}{K^2}$, then $I-\lambda_1A_1$ is nonexpansive. Similarly, we can show that $I-\lambda_2A_2$ is nonexpansive.

(2) Let $\{x_n\}$ and $\{y_n\}$ be the sequences defined as in Theorem 3.1. From

Lemma 2.1, nonexpansiveness of S, Q_C and the convexity of $\|.\|^2$, we obtain

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \le a_n\|f(x_n) - x^*\|^2 + b_n\|x_n - x^*\|^2 + c_n\|t_n - x^*\|^2 \\
&\le a_n\|f(x_n) - x^*\|^2 + b_n\|x_n - x^*\|^2 + c_n[\|y_n - y^* - \lambda_1(A_1y_n - A_1y^*)\|^2] \\
&\le a_n\|f(x_n) - x^*\|^2 + b_n\|x_n - x^*\|^2 \\
&\quad + c_n[\|y_n - y^*\|^2 - 2\lambda_1\langle A_1y_n - A_1y^*, j(y_n - y^*)\rangle + 2K^2{\lambda_1}^2\|A_1y_n - A_1y^*\|^2] \\
&\le a_n\|f(x_n) - x^*\|^2 + b_n\|x_n - x^*\|^2 \\
&\quad + c_n[\|y_n - y^*\|^2 - 2\lambda_1\alpha_1\|A_1y_n - A_1y^*\|^2 + 2{\lambda_1}^2K^2\|A_1y_n - A_1y^*\|^2] \\
&= a_n\|f(x_n) - x^*\|^2 + b_n\|x_n - x^*\|^2 + c_n\|y_n - y^*\|^2 \\
&\quad - 2c_n\lambda_1(\alpha_1 - \lambda_1K^2)\|A_1y_n - A_1y^*\|^2 \\
&\le a_n\|f(x_n) - x^*\|^2 + b_n\|x_n - x^*\|^2 + c_n\|x_n - x^* - \lambda_2(A_2x_n - A_2x^*)\|^2 \\
&\quad - 2c_n\lambda_1(\alpha_1 - \lambda_1K^2)\|A_1y_n - A_1y^*\|^2 \\
&\le a_n\|f(x_n) - x^*\|^2 + b_n\|x_n - x^*\|^2 \\
&\quad + c_n[\|x_n - x^*\|^2 - 2\lambda_2\langle A_2x_n - A_2x^* j(x_n - x^*)\rangle + 2{\lambda_2}^2K^2\|A_2x_n - A_2x^*\|^2] \\
&\quad - 2c_n\lambda_1(\alpha_1 - \lambda_1K^2)\|A_1y_n - A_1y^*\|^2 \\
&\le a_n\|f(x_n) - x^*\|^2 + b_n\|x_n - x^*\|^2 \\
&\quad + c_n[\|x_n - x^*\|^2 - 2\lambda_2\alpha_2\|A_2x_n - A_2x^*\|^2 + 2{\lambda_2}^2K^2\|A_2x_n - A_2x^*\|^2] \\
&\quad - 2c_n\lambda_1(\alpha_1 - \lambda_1K^2)\|A_1y_n - A_1y^*\|^2 \\
&= a_n\|f(x_n) - x^*\|^2 + (1 - a_n)\|x_n - x^*\|^2 - 2c_n\lambda_1(\alpha_1 - \lambda_1K^2)\|A_1y_n - A_1y^*\|^2 \\
&\quad - 2c_n\lambda_2(\alpha_2 - \lambda_2K^2)\|A_2x_n - A_2x^*\|^2.
\end{aligned}$$

Which implies that

$$\begin{aligned}
&2c_n\lambda_1(\alpha_1 - \lambda_1K^2)\|A_1y_n - A_1y^*\|^2 + 2c_n\lambda_2(\alpha_2 - \lambda_2K^2)\|A_2x_n - A_2x^*\|^2 \\
&\le a_n\|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\le a_n\|f(x_n) - x^*\|^2 + \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|).
\end{aligned}$$

From (i)-(iii) and (3.4), we obtain $\|A_1y_n - A_1y^*\| \to 0$ and $\|A_2x_n - A_2x^*\| \to 0$ as $n \to \infty$. $\square$

Let X be a uniformly convex and 2-uniformly smooth Banach space with the 2-uniformly smooth constant K, let C be a nonempty closed convex subset of X. Let $A_1 : C \to X$ be relaxed (c_1^*, d_1^*)-cocoercive and L_1-Lipschitzian and $A_2 : C \to X$ be relaxed (c_2^*, d_2^*)-cocoercive and L_2-Lipschitzian. If $0 < \lambda_1 < \frac{d_1^* - c_1^*{L_1}^2}{K^2{L_1}^2}$ and $0 < \lambda_2 < \frac{d_2^* - c_2^*{L_2}^2}{K^2{L_2}^2}$, then we have

(1) $I - \lambda_1A_1$ and $I - \lambda_2A_2$ are nonexpansive and

(2) $\|A_1y_n - A_1y^*\| \to 0$ and $\|A_2x_n - A_2x^*\| \to 0$ as $n \to \infty$ for all $x^* \in \Omega$ and $y^* = Q_C(x^* - \lambda_2A_2x^*)$ where $\{x_n\}$ and $\{y_n\}$ are two sequences defined as in Theorem 3.1.

Proof. (1) For any $x, y \in C$, it follows from Lemma 2.1 that

$$\begin{aligned}
&\|(I-\lambda_1 A_1)x-(I-\lambda_1 A_1)y\|^2 = \|x-y-\lambda_1(A_1x-A_1y)\|^2\\
&\le \|x-y\|^2-2\lambda_1\langle A_1x-A_1y, j(x-y)\rangle+2{\lambda_1}^2K^2\|A_1x-A_1y\|^2\\
&\le \|x-y\|^2-2\lambda_1(-c_1^*\|A_1x-A_1y\|^2+d_1^*\|x-y\|^2)+2{\lambda_1}^2K^2\|A_1x-A_1y\|^2\\
&\le \|x-y\|^2+2(\lambda_1c_1^*{L_1}^2-\lambda_1d_1^*+K^2{\lambda_1}^2{L_1}^2)\|x-y\|^2.
\end{aligned}$$

It clear that if $0<\lambda_1<\frac{d_1^*-c_1^*{L_1}^2}{K^2{L_1}^2}$, then $I-\lambda_1A_1$ is nonexpansive. Similarly, we can show that $I-\lambda_2A_2$ is nonexpansive.

(2) Let $\{x_n\}$ and $\{y_n\}$ be the sequences defined as in Theorem 3.1. From Lemma 2.1, nonexpansiveness of S, Q_C and the convexity of $\|.\|^2$, we obtain

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2 \le a_n\|f(x_n)-x^*\|^2+b_n\|x_n-x^*\|^2+c_n\|t_n-x^*\|^2\\
&\le a_n\|f(x_n)-x^*\|^2+b_n\|x_n-x^*\|^2+c_n[\|y_n-y^*-\lambda_1(A_1y_n-A_1y^*)\|^2]\\
&\le a_n\|f(x_n)-x^*\|^2+b_n\|x_n-x^*\|^2\\
&\quad+c_n[\|y_n-y^*\|^2-2\lambda_1\langle A_1y_n-A_1y^*, j(y_n-y^*)\rangle+2K^2{\lambda_1}^2\|A_1y_n-A_1y^*\|^2]\\
&\le a_n\|f(x_n)-x^*\|^2+b_n\|x_n-x^*\|^2\\
&\quad+c_n[\|y_n-y^*\|^2-2\lambda_1(-c_1^*\|A_1y_n-A_1y^*\|^2+d_1^*\|y_n-y^*\|^2)\\
&\quad+2{\lambda_1}^2K^2\|A_1y_n-A_1y^*\|^2]\\
&\le a_n\|f(x_n)-x^*\|^2+b_n\|x_n-x^*\|^2\\
&\quad+c_n[\|y_n-y^*\|^2+2\lambda_1c_1^*\|A_1y_n-A_1y^*\|^2-\frac{2\lambda_1d_1^*}{{L_1}^2}\|A_1y_n-A_1y^*\|^2\\
&\quad+2{\lambda_1}^2K^2\|A_1y_n-A_1y^*\|^2]\\
&= a_n\|f(x_n)-x^*\|^2+b_n\|x_n-x^*\|^2+c_n\|y_n-y^*\|^2\\
&\quad-2c_n\lambda_1(\frac{d_1^*}{{L_1}^2}-c_1^*-\lambda_1K^2)\|A_1y_n-A_1y^*\|^2\\
&\le a_n\|f(x_n)-x^*\|^2+b_n\|x_n-x^*\|^2+c_n\|x_n-x^*-\lambda_2(A_2x_n-A_2x^*)\|^2\\
&\quad-2c_n\lambda_1(\frac{d_1^*}{{L_1}^2}-c_1^*-\lambda_1K^2)\|A_1y_n-A_1y^*\|^2\\
&\le a_n\|f(x_n)-x^*\|^2+b_n\|x_n-x^*\|^2\\
&\quad+c_n[\|x_n-x^*\|^2-2\lambda_2\langle A_2x_n-A_2x^*, j(x_n-x^*)\rangle
\end{aligned}$$

$$
\begin{aligned}
&\quad + 2K^2{\lambda_2}^2\|A_2x_n - A_2x^*\|^2]\\
&\quad - 2c_n\lambda_1(\frac{d_1^*}{{L_1}^2} - c_1^* - \lambda_1K^2)\|A_1y_n - A_1y^*\|^2\\
&\le a_n\|f(x_n) - x^*\|^2 + b_n\|x_n - x^*\|^2\\
&\quad + c_n[\|x_n - x^*\|^2 - 2\lambda_2(-c_2^*\|A_2x_n - A_2x^*\|^2 + d_2^*\|x_n - x^*\|^2)\\
&\quad + 2K^2{\lambda_2}^2\|A_2x_n - A_2x^*\|^2]\\
&\quad - 2c_n\lambda_1(\frac{d_1^*}{{L_1}^2} - c_1^* - \lambda_1K^2)\|A_1y_n - A_1y^*\|^2\\
&\le a_n\|f(x_n) - x^*\|^2 + b_n\|x_n - x^*\|^2\\
&\quad + c_n[\|x_n - x^*\|^2 + 2\lambda_2c_2^*\|A_2x_n - A_2x^*\|^2 - \frac{2\lambda_2d_2^*}{{L_2}^2}\|A_2x_n - A_2x^*\|^2\\
&\quad + 2K^2{\lambda_2}^2\|A_2x_n - A_2x^*\|^2]\\
&\quad - 2c_n\lambda_1(\frac{d_1^*}{{L_1}^2} - c_1^* - \lambda_1K^2)\|A_1y_n - A_1y^*\|^2\\
&= a_n\|f(x_n) - x^*\|^2 + (1 - a_n)\|x_n - x^*\|^2\\
&\quad - 2c_n\lambda_1(\frac{d_1^*}{{L_1}^2} - c_1^* - \lambda_1K^2)\|A_1y_n - A_1y^*\|^2\\
&\quad - 2c_n\lambda_2(\frac{d_2^*}{{L_2}^2} - c_2^* - \lambda_2K^2)\|A_2x_n - A_2x^*\|^2.
\end{aligned}
$$

Which implies that

$$
\begin{aligned}
&2c_n\lambda_1(\frac{d_1^*}{{L_1}^2} - c_1^* - \lambda_1K^2)\|A_1y_n - A_1y^*\|^2\\
&\ + 2c_n\lambda_2(\frac{d_2^*}{{L_2}^2} - c_2^* - \lambda_2K^2)\|A_2x_n - A_2x^*\|^2\\
&\ \le a_n\|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2\\
&\ \le a_n\|f(x_n) - x^*\|^2 + \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|).
\end{aligned}
$$

From (i)-(iii) and (3.4), we obtain $\|A_1y_n - A_1y^*\| \to 0$ and $\|A_2x_n - A_2x^*\| \to 0$ as $n \to \infty$. $\square$

By using the same proof as in Example 3 and Example 3, we obtain the following example.

Let X be a uniformly convex and 2-uniformly smooth Banach space with the 2-uniformly smooth constant K, let C be a nonempty closed convex subset of X. Let $A_1 : C \to X$ be $\alpha-$inverse strongly accretive and $A_2 : C \to X$ be relaxed (c, d)-cocoercive and $L-$Lipschitzian. If $0 < \lambda_1 < \frac{\alpha}{K^2}$ and $0 < \lambda_2 < \frac{d-cL^2}{K^2L^2}$, then we have

(1) $I - \lambda_1A_1$ and $I - \lambda_2A_2$ are nonexpansive and

(2) $\|A_1y_n - A_1y^*\| \to 0$ and $\|A_2x_n - A_2x^*\| \to 0$ as $n \to \infty$ for all $x^* \in \Omega$

and $y^* = Q_C(x^* - \lambda_2 A_2 x^*)$ where $\{x_n\}$ and $\{y_n\}$ are two sequences defined as in Theorem 3.1.

Let $\mathcal{A}$ be the class of all α_1-inverse-strongly accretive mappings from C into X, $\mathcal{B}$ the class of all α_2-inverse-strongly accretive mappings from C into X, $\mathcal{C}$ the class of all L-Lipschitzian and relaxed (c, d)-cocoercive mappings from C into X and $\mathcal{D}$ the class of all L_1-Lipschitzian and relaxed (c^*, d^*)-cocoercive mappings from C into X.

From Theorem 3.1, Example 3 - 3, we obtain the following result.

Corollary 3.2. *Let X be a uniformly convex and 2-uniformly smooth Banach space with the 2-uniformly smooth constant K, let C be a nonempty closed convex subset of X and Q_C a sunny nonexpansive retraction from X onto C. Let $A_1, A_2 : C \to X$ be two mappings satisfying one of the following conditions:*

(1) $A_1 \in \mathcal{A}$, $A_2 \in \mathcal{B}$, $0 < \lambda_1 < \frac{\alpha_1}{K^2}$ *and* $0 < \lambda_2 < \frac{\alpha_2}{K^2}$;

(2) $A_1 \in \mathcal{C}$, $A_2 \in \mathcal{D}$, $0 < \lambda_1 < \frac{d-cL^2}{K^2L^2}$ *and* $0 < \lambda_2 < \frac{d^*-c^*{L_1}^2}{K^2{L_1}^2}$;

(3) $A_1 \in \mathcal{A}$, $A_2 \in \mathcal{C}$, $0 < \lambda_1 < \frac{\alpha_1}{K^2}$ *and* $0 < \lambda_1 < \frac{d-cL^2}{K^2L^2}$.

Let f be a contractive mapping with the constant $\alpha \in (0, 1)$ and $S : C \to C$ a nonexpansive mapping such that $\Omega = F(S) \bigcap F(G) \neq \emptyset$, where G is the mapping defined as in Lemma 2.10. For a given $x_1 \in C$, let $\{x_n\}$ and $\{y_n\}$ be the sequences generated by

$$\begin{cases} y_n = Q_C(x_n - \lambda_2 A_2 x_n), \\ x_{n+1} = a_n f(x_n) + b_n x_n + c_n S Q_C(y_n - \lambda_1 A_1 y_n), \quad n \geq 1, \end{cases}$$

where $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are three sequences in $(0, 1)$. If the conditions (i)-(iii) in Theorem 3.1 hold, then $\{x_n\}$ converges strongly to $q \in \Omega$, which solves the following variational inequality:

$$\langle q - f(q), j(q - p)\rangle \leq 0, \quad \forall f \in \Pi_C, \ p \in \Omega.$$

Remark 3.1. Corollary 3.2 improves and extends Theorem 3.4 of Katchang and Kumam [9] and Theorem 3.1 of Yao et al. [8].

4 Applications

Using Corollary 3.2, we prove two theorems in a real Banach space.

In a real Banach space X, we recall that an accretive operator T is m-accretive if $R(I + rT) = X$ for all $r > 0$, where I is the identity operator. The set of zero of T is denoted by $T^{-1}(0)$, that $T^{-1}(0) = \{z \in D(T) : 0 \in T(z)\}$. We denote the resolvent of T by $J_r^T = (I + rT)^{-1}$ for each $r > 0$, it is known that if T is m-accretive then $J_r^T : X \to X$ is nonexpansive and $F(J_r^T) = T^{-1}(0)$ for each $r > 0$.

Theorem 4.1. *Let X be a uniformly convex and 2-uniformly smooth Banach space with the 2-uniformly smooth constant K. Let $A : X \to X$ be an β-inverse-strongly accretive mapping with $0 < \lambda < \frac{\beta}{K^2}$ and f be a contraction of E into itself with the constant $\alpha \in (0, 1)$. Let T be an m-accretive mapping such that $\Omega = A^{-1}(0) \bigcap T^{-1}(0) \neq \emptyset$. For a given $x_1 \in C$, let $\{x_n\}$ and $\{y_n\}$ be the sequences generated by*

$$\begin{cases} y_n = x_n - \lambda A x_n, \\ x_{n+1} = a_n f(x_n) + b_n x_n + c_n J_r^T (y_n - \lambda A y_n), \quad n \geq 1, \end{cases}$$

where $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are three sequences in $(0, 1)$. If the conditions (i)-(iii) in Theorem 3.1 hold, then $\{x_n\}$ converges strongly to $q \in \Omega$, which solves the following variational inequality:

$$\langle q - f(q), j(q - p)\rangle \leq 0, \quad \forall f \in \Pi_C, \ p \in \Omega.$$

Proof. We have $\lambda = \lambda_1 = \lambda_2$,$A = A_1 = A_2$, $C = X$ and $Q_X = I$. In this case, we have $A^{-1}(0) = F(I - \lambda A) = S(X, A)$ (see [4]). We want to show that $S(X, A) = F(G)$. Indeed, it is sufficient to show that $F(G) \subseteq S(X, A)$. Let $x^* \in F(G)$, then $x^* = y^* - \lambda A y^*$, where $y^* = x^* - \lambda A x^*$. We claim that $x^* = y^*$. Assume that $x^* \neq y^*$, therefore $Ax^* \neq 0$, $Ay^* \neq 0$ and $Ay^* - Ax^* \neq 0$. It follows from Example 3(1) that

$$\begin{aligned} \|x^* - y^*\|^2 &= \|(I - \lambda A)y^* - (I - \lambda A)x^*\|^2 \\ &\leq \|y^* - x^*\|^2 + 2\lambda(\lambda K^2 - \beta)\|Ay^* - Ax^*\|^2 < \|y^* - x^*\|^2, \end{aligned}$$

which hence leads to a contradiction. This show that $x^* = y^*$, therefore $x^* \in F(I - \lambda A) = S(X, A) = A^{-1}(0)$. Thus, by Corollary 3.2, we obtain the desired result. □

Let C be a nonempty closed convex subset of X. A mapping $T : C \to C$ k-strictly pseudocontractive (see [22]) if for each $x, y \in C$, there exists a constant $k > 0$ and $j(x, y) \in J(x, y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 - k\|(I - T)x - (I - T)y\|^2. \tag{4.1}$$

It is clear that (4.1) is equivalent to the following:

$$\langle (I - T)x - (I - T)y, j(x - y)\rangle \geq k\|(I - T)x - (I - T)y\|^2.$$

Hence, if T is k-strictly pseudocontractive then $(I - T)$ is k-inverse-strongly accretive mapping.

Theorem 4.2. *Let X be a uniformly convex and 2-uniformly smooth Banach space with the 2-uniformly smooth constant K, let C be a nonempty closed* *convex subset and a sunny nonexpansive retraction of X. Let the mappings $T_1, T_2 : C \to C$ be k_1-strictly pseudocontractive and k_2-strictly pseudocontractive with $0 < \lambda_1 < \frac{k_1}{K^2}$ and $0 < \lambda_2 < \frac{k_2}{K^2}$, respectively. Let f be a contractive mapping with the constant $\alpha \in (0, 1)$ and $S : C \to C$ a nonexpansive mapping such that $\Omega = F(S) \bigcap F(G) \neq \emptyset$, where G is the mapping defined as in Lemma 2.10. For a given $x_1 \in C$, let $\{x_n\}$ and $\{y_n\}$ be the sequences generated by*

$$\begin{cases} y_n = (1-\lambda_2)x_n + \lambda_2 T_2 x_n, \\ x_{n+1} = a_n f(x_n) + b_n x_n + c_n S((1-\lambda_1)y_n + \lambda_1 T_1 y_n), \quad n \geq 1, \end{cases}$$

where $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are three sequences in $(0, 1)$. If the conditions (i)-(iii) in Theorem 3.1 hold, then $\{x_n\}$ converges strongly to $q \in \Omega$, which solves the following variational inequality:

$$\langle q - f(q), j(q-p)\rangle \leq 0, \quad \forall f \in \Pi_C, \ p \in \Omega.$$

Proof. Let $A_1 = I - T_1$ and $A_2 = I - T_2$, then A_1 is k_1-strictly pseudocontractive and A_2 is k_2-strictly pseudocontractive, respectively. Since C is a sunny nonexpansive retraction of X, there exists a sunny nonexpansive retraction Q_C such that

$$Q_C(x_n - \lambda_2 A_2 x_n) = x_n - \lambda_2 A_2 x_n = (1-\lambda_2)x_n + \lambda_2 T_2 x_n$$

and

$$Q_C(y_n - \lambda_1 A_1 y_n) = y_n - \lambda_1 A_1 y_n = (1-\lambda_1)y_n + \lambda_1 T_1 y_n.$$

Therefore, the conclusion follows immediately from Corollary 3.2. □

Acknowledgement. The first author was supported by the Commission on Higher Education, the Thailand Research Fund and Thaksin University under grant MRG5680072.

References

[1] X. Qin, M. Shang and Y. Su, Strong convergence of a general iterative algorithm for equilibrium problems and variational inequality problems, *Math. Comput. Modelling*, **48** (2008), 1033-1046. http://dx.doi.org/10.1016/j.mcm.2007.12.008

[2] Y. Shehu, Iterative method for fixed point problem, variational inequality and generalized mixed equilibrium problems with applications, *J. Glob. Optim.*, **52** (2012), 57-77. http://dx.doi.org/10.1007/s10898-011-9679-0

[3] R. Wangkeeree and P. Preechasilp, A new iterative scheme for solving the equilibrium problems, variational inequality problems, and fixed point problems in Hilbert spaces, *J. Appl. Math.*, **2012** (2012), 1-21. http://dx.doi.org/10.1155/2012/154968

[4] K. Aoyama, H. Iiduka and W. Takahashi, Weak convergence of an iterative sequence for accretive operators in Banach spaces, *Fixed Point Theory Appl.*, **2006** (2006), 1-13. http://dx.doi.org/10.1155/fpta/2006/35390

[5] A. Moudafi and Z. Huang, About the relaxed cocoercivity and the convergence of the proximal point algorithm, *J. Egypt. Math. Soc.*, **21** (2013), 281-284. http://dx.doi.org/10.1016/j.joems.2013.03.014

[6] Y. Qing and S. Y. Cho, Proximal point algorithms for zero points of nonlinear operators, *Fixed Point Theory and Appl.*, **2014** (2014), 42. http://dx.doi.org/10.1186/1687-1812-2014-42

[7] Y. Tang, Z. Bao and D. Wen, An algorithm with general errors for the zero point of monotone mappings in Banach spaces, *J. Inequal. Appl.*, **2014** (2014), 484. http://dx.doi.org/10.1186/1029-242x-2014-484

[8] Y. Yao, M. A. Noor, K. I. Noor, Y. C. Liou and H. Yaqoob, Modified extragradient method for a system of variational inequalities in Banach spaces, *Acta Appl. Math.*, **110** (2010), 1211-1224. http://dx.doi.org/10.1007/s10440-009-9502-9

[9] P. Katchang and P. Kumam, Convergence of iterative algorithm for finding common solution of fixed points and general system of variational inequalities for two accretive operators, *Thai J. Math.*, **9** (2011), 343-360.

[10] G. Cai and S. Bu, Strong convergence theorems based on a new modified extragradient method for variational inequality problems and fixed point problems in Banach spaces, *Comput. Math. Appl.*, **62** (2011), 2567-2579. http://dx.doi.org/10.1016/j.camwa.2011.07.056

[11] G. Cai and S. Bu, Convergence analysis for variational inequality problems and fixed point problems in 2-uniformly smooth and uniformly convex Banach spaces, *Math. Comput. Modelling*, **55** (2012), 538-546. http://dx.doi.org/10.1016/j.mcm.2011.08.031

[12] P. Katchang and P. Kumam, An iterative algorithm for finding a common solution of fixed points and a general system of variational inequalities for two inverse strongly accretive operators, *Positivity*, **15** (2011), 281-295. http://dx.doi.org/10.1007/s11117-010-0074-8

[13] X. Qin, S. Y. Cho and S. M. Kang, Convergence of an iterative algorithm for systems of variational inequalities and nonexpansive mappings with applications, *J. Comput. Appl. Math.*, **233** (2009), 231-240. http://dx.doi.org/10.1016/j.cam.2009.07.018

[14] H. K. Xu, Inequalities in Banach spaces with applications, *Nonlinear Anal.*, **16** (1991), 1127-1138. http://dx.doi.org/10.1016/0362-546x(91)90200-k

[15] S. S. Chang, On Chidumes open questions and approximate solutions of multivalued strongly accretive mapping equations in Banach spaces, *J. Math. Anal. Appl.*, **216** (1997), 94-111. http://dx.doi.org/10.1006/jmaa.1997.5661

[16] H. K. Xu, Viscosity approximation methods for nonexpansive mappings, *J. Math. Anal. Appl.*, **298** (2004), 279-291. http://dx.doi.org/10.1016/j.jmaa.2004.04.059

[17] S. Reich, Asymptotic behavior of contractions in Banach spaces, *J. Math. Anal. Appl.*, **44** (1973), 57-70. http://dx.doi.org/10.1016/0022-247x(73)90024-3

[18] S. Kitahara and W. Takahashi, Image recovery by convex combinations of sunny nonexpansive retractions, *Topol. Methods Nonlinear Anal.*, **2** (1993), 333-342.

[19] T. Suzuki, Strong convergence of Krasnoselskii and Manns type sequences for one-parameter nonexpansive semigroups without Bochner integrals, *J. Math. Anal. Appl.*, **305** (2005), 227-239. http://dx.doi.org/10.1016/j.jmaa.2004.11.017

[20] R. E. Bruck, Properties of fixed point sets of nonexpansive mappings in Banach spaces, *Trans. Amer. Math. Soc.*, **179** (1973), 251-262. http://dx.doi.org/10.1090/s0002-9947-1973-0324491-8

[21] S. Kamimura and W. Takahashi, Strong convergence of a proximal-type algorithm in Banach space, *SIAM J. Optim.*, **12** (2002), 938-945. http://dx.doi.org/10.1137/s105262340139611x

[22] F. E. Browder and W. V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert space, *J. Math. Anal. Appl.*, **20** (1967), 197-228. http://dx.doi.org/10.1016/0022-247x(67)90085-6

Received: July 7, 2016; Published: August 28, 2016