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An Iterative Method for Solving Delay Differential Equations Applied to Biological Models

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Abstract

In this work, we apply the Zhou's method or Differential Transformation Method (DTM) for solving some models that arises in biological sciences, which are nonlinear delay differential equations. The efficiency of DTM is illustrated by investigating the convergence results on numerical models that show the reliability and accuracy of this method.

Keywords: Differencial transformation, Delay differential equations, Spread of infections, Growth of tumors in mice

1 Introduction

The Zhou's method or Differential Transformation Method (DTM) is a semi-analytical-numerical technique for solving differential equations (ODEs and PDEs). This method has been studied since it can be applied to nonlinear differential equations without requiring discretization or perturbation. Delay Differential Equations (DDEs) arise in biological models among others and many numerical methods were developed this type of differential equations such as the Adomian Decomposition Method, Homotopy Perturbation Method and Homotopy Analysis Method. In this paper, we show superiority of the Zhou's method by applying them on the some type DDEs. The power series solution of the reduced equations transforms into an approximate implicit solution of the original problem.

2 The Zhou's Method

Differential transformation method of the function $y(x)$ is defined as

$$Y(k) = \frac{1}{k!}\left(\frac{d^k y(x)}{dx^k}\right)_{x=x_0} \tag{1}$$

where $Y(k)$ is the transformed function and $y(x)$ is the original function. Here, the inverse differential transformation is defined by

$$y(x) = \sum_{k=0}^{\infty} Y(k)x^k \tag{2}$$

The following theorems can be deduced from equations (1) and (2).

Theorem 2.1. *If $f(x) = g(x) \pm h(x)$, then $F(k) = G(k) \pm H(k)$*

Theorem 2.2. *If $f(x) = \alpha g(x)$, then $F(k) = \alpha G(k) \pm H(k)$, with $\alpha \in \mathbb{R}$.*

Theorem 2.3. *If $f(x) = \frac{dy}{dx}$, then $F(k) = (k+1)Y(k+1)$.*

Theorem 2.4. *If $f(x) = \frac{d^n y}{dx^n}$, then $F(k) = (k+1)(k+2)\cdots(k+n)Y(k+n)$.*

Theorem 2.5. *If $f(x) = g(x)h(x)$, then $F(k) = \sum_{r=0}^{k} G(k)H(k-r)$.*

Theorem 2.6. *If $f(x) = x^n$, then $F(k) = \delta(k-n)$, where δ is the Dirac's delta function.*

Theorem 2.7. *(Cárdenas). If $f(x) = x^m f(x)$, then*

$$F(k) = \begin{cases} 0, & k < m \\ Y(k-m), & k \geq m \end{cases}$$

3 The Zhou's method for DDEs

We consider the $n-$order delay differential equation

$$y^{(n)}(x) = f(x, y(x), y(\alpha_1(x)), \dots, y(\alpha_m(x))) \tag{3}$$

with $x \in [0, b]$ and subject to the conditions

$$\big(y(x), y'(x), \dots, y^{(n-1)}(x)\big) = (h_1(x), h_2(x), \dots, h_n(x))$$

where $x \in [a, 0]$. Here, a is the minimum of the values $\alpha_i(x) \leq x$ for all $x \in [0, b]$ and $i \in [1, m]$. We assume that the delay functions $\alpha_i(x)$ and $g_i(x)$ are sufficiently smooth.

3.1 Extension of the DTM to delay equations

In this section we state the fundamental theorems and corollaries of this work. Assume that $F(k)$ and $Y(k)$ are the differential transformations of the functions $f(x)$ and $y(x)$ respectively and $\alpha, \beta \in]0, 1[$. Then

Theorem 3.1. *If $f(x) = y(\alpha x)$, then $F(k) = \alpha^k Y(k)$.*

Corollary 3.2. *If $f(x) = y(\frac{\alpha}{x})$, then $F(k) = \frac{1}{\alpha^k} Y(k)$, with $\alpha \geq 1$.*

Theorem 3.3. *If $f(x) = y_1(\alpha x) y_2(\beta x)$, then*

$$F(k) = \sum_{l=0}^{k} \alpha^l \beta^{k-l} Y_1(l) Y_2(k-l)$$

with $\alpha, \beta \geq 1$.

Corollary 3.4. *If $f(x) = y_1(\frac{\alpha}{x}) y_2(\frac{\beta}{x})$, then*

$$F(k) = \sum_{l=0}^{k} \frac{1}{\alpha^l} \frac{1}{\beta^{k-l}} Y_1(l) Y_2(k-l)$$

with $\alpha, \beta \geq 1$.

Theorem 3.5. *If $f(x) = \dfrac{d^n}{dx^n} y(\alpha x)$, then*

$$F(k) = \frac{(k+n)!}{k!} \alpha^{k+n} Y(k+n)$$

Corollary 3.6. *If* $f(x) = \dfrac{d^n}{dx^n} y(x-\alpha)$ *with* $\alpha > 0$, *then*

$$F(k) = \frac{(k+n)!}{k!} \sum_{i=k+n}^{M} (-\alpha)^{i-k-n} \binom{i}{k+n} Y(i)$$

to $M \to \infty$.

Theorem 3.7. *If* $f(x) = \int_0^{mx} y(\alpha s)\, ds$, *then*

$$F(k) = \frac{1}{k} m^k \alpha^{k-1} Y(k-1)$$

Corollary 3.8. *If* $f(x) = \int_0^{mx} y_1(\alpha s) y_2(\beta s)\, ds$, *then*

$$F(k) = \frac{1}{k} \sum_{l=0}^{k-1} m^k \alpha^l \beta^{k-l-1} Y_1(l) Y_2(k-l-1)$$

Theorem 3.9. *If* $f(x) = g(x-\alpha)$ *with* $\alpha > 0$, *then*

$$F(k) = \sum_{i=k}^{N} (-\alpha)^{i-k} \binom{i}{k} G(i),$$

to $N \to \infty$.

Theorem 3.10. *(Cárdenas). If* $f(x) = g^2(x-\alpha)$ *with* $\alpha > 0$, *then*

$$F(k) = \sum_{l=0}^{k} \left[\sum_{i=l}^{N} (-\alpha)^{i-l} \binom{i}{l} G(i) \cdot \sum_{h_1=k-l}^{N} (-\alpha)^{h_1-k+l} \binom{h_1}{k-l} G(h_1) \right]$$

Corollary 3.11. *If* $f(x) = y_1(\alpha x) y_2(x-\beta)$, *then*

$$F(k) = \sum_{l=0}^{k} \sum_{h_1=k-l}^{N} \alpha^l (-\beta)^{h_1-k+l} \binom{h_1}{k-l} Y_1(l) Y_2(h_1)$$

Corollary 3.12. *If* $f(x) = y_1(x-\alpha) y_2(x-\beta)$, *then*

$$F(k) = \sum_{l=0}^{k} \left[\sum_{i=l}^{N} (-\alpha)^{i-l} \binom{i}{l} G(i) \cdot \sum_{h_1=k-l}^{N} (-\beta)^{h_1-k+l} \binom{h_1}{k-l} G(h_1) \right]$$

4 Numerical models

In order to illustrate the advantages and the accuracy of the DTM for solving nonlinear delay differential equations, we have applied the method to different delay models.

4.1 Model for growth of tumors in mice

Tumor growth of *Ehrlich ascities* in mice is modeled by the following initial value problem with delay

$$\begin{cases} y'(t) = ry(t-\alpha)\left(1 - \dfrac{y(t-\alpha)}{C}\right) \\ y(t) = y_0(t) \geq 0 \end{cases} \tag{4}$$

for all $t \in]-\alpha, 0[$. Here, $y(t)$ is related to the number of cells (concentration) in the mouse; r is the rate of proportionality (net) of reproduction of tumor cells; C is the storage capacity and is the delay shows the duration of a cycle multiplicity of cells. For instance, we consider the following particular case of the model (4)

$$\begin{cases} y'(t) = 0.1y\left(\dfrac{t}{2}\right)\left(1 - y\left(\dfrac{t}{2}\right)\right) \\ y(0) = 0.1 \end{cases} \tag{5}$$

We can see that the differential equation associated with this problem can be written so that the DTM theorems for equations with delay are easy to apply, that is to say,

$$y(t) = 0.1y\left(\frac{t}{2}\right) - 0.1y\left(\frac{t}{2}\right)y\left(\frac{t}{2}\right)$$

Now, applying DTM we arrived to the recurrence equation

$$\frac{(k+1)!}{k!}Y(k+1) = 0.1\frac{1}{2^k}Y(k) - 0.1\sum_{l=0}^{k}\frac{1}{2^l}\frac{1}{2^{k-l}}Y(l)Y(k-l)$$

or

$$Y(k+1) = \frac{k!}{(k+1)!}\left[\frac{0.1}{2^k}Y(k) - 0.1\sum_{l=0}^{k}\frac{1}{2^k}Y(l)Y(k-l)\right] \tag{6}$$

From the initial condition $y(0) = 0.1$ we have that $Y(0) = 0.1$. Now, solving recurrence equation (6) we obtain:

If $k = 0$, then

$$\begin{aligned} Y(1) &= \frac{0!}{1!}\left[\frac{0.1}{2^0}Y(0) - 0.1\left(\frac{1}{2^0}Y(0)Y(0)\right)\right] \\ &= 1\left[0.1Y(0) - 0.1Y(0)Y(0)\right] = (0.1)(0.1) - (0.1)^3 = \boxed{0.009} \end{aligned}$$

If $k = 1$, then

$$\begin{aligned} Y(2) &= \frac{1!}{2!}\left[\frac{0.1}{2^1}Y(1) - 0.1\left(\frac{1}{2^1}Y(0)Y(1) + \frac{1}{2^1}Y(1)Y(0)\right)\right] \\ &= \frac{1}{2}\left[0.05Y(1) - 0.1\left(\frac{1}{2}Y(0)Y(1) + \frac{1}{2}Y(1)Y(0)\right)\right] \\ &= \frac{1}{2}\left[0.05(0.009) - 0.1\left(0.5(0.1)(0.009) + 0.5(0.009)(0.1)\right)\right] = \boxed{0.00018} \end{aligned}$$

If $k = 2$, then

$$\begin{aligned} Y(3) &= \frac{2!}{3!}\left[\frac{0.1}{2^2}Y(2) - \frac{0.1}{2^2}\left(Y(0)Y(2) + Y(1)Y(1) + Y(2)Y(0)\right)\right] \\ &= \frac{1}{3}\left[\frac{0.1}{4}Y(2) - \frac{0.1}{4}\left(Y(0)Y(2) + Y(1)Y(1) + Y(2)Y(0)\right)\right] \\ &= \frac{1}{3}\left[0.025(0.00018) - \frac{0.1}{4}\left(0.1(0.00018) + (0.009)^2 + (0.00018)(0.1)\right)\right] \\ &= \boxed{-0.00333416} \end{aligned}$$

and so on. Therefore, using equation (2) we arrived

$$\begin{aligned} y(x) &= Y(0)x^0 + Y(1)x^1 + Y(2)x^2 + Y(3)x^3 + Y(4)x^4 + \dots \\ &= 0.1 + 0.009x + 0.00018x^2 - 0.00333416x^3 - 0.00000928326x^4 + \dots \end{aligned}$$

5 Spread of infections

According *Kermack-McKendrick model*, the number of new infections per unit time is proportional to the product and $y_1(t)$ and $y_2(t)$, where $y_1(t)$ is the fraction of susceptible study population; $y_2(t)$ is the infected and $y_3(t)$ represents the fraction immunized.

Here, for convenience we assume that all the constants of proportionality are equal to 1. Similarly we can assume that the immunized population is again susceptible after a fixed period of time we will call α_1. So, just as we can introducing an incubation period α_2 obtaining model

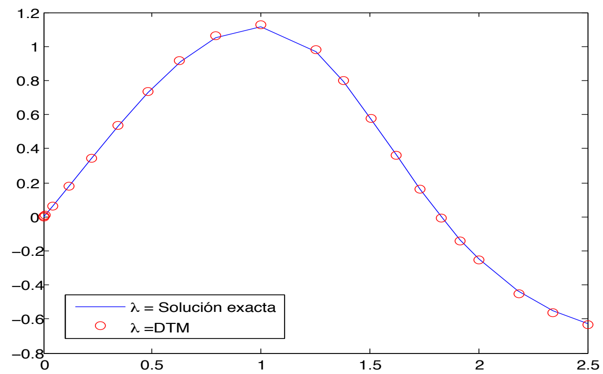

Figure 1: Model for growth of tumors in mice

$$\begin{cases} y_1'(t) & = -y_1(t)y_2(t-\alpha_1) + y_2(t-\alpha_2) \\ y_2'(t) & = y_1(t)y_2(t-\alpha_1) - y_2(t) \\ y_3'(t) & = y_2(t) - y_2(t-\alpha_2) \end{cases} \tag{7}$$

We consider a special case of the model (7)

$$\begin{cases} y_1'(t) & = -y_1(t)y_2(t-1) + y_2(t-10) \\ y_2'(t) & = y_1(t)y_2(t-1) - y_2(t) \\ y_3'(t) & = y_2(t) - y_2(t-10) \end{cases} \tag{8}$$

subject to the initial conditions $y_1(0) = 0$, $y_2(0) = 1$ and $y_3(0) = 5$. Applying the differential transformation method (extension of DTM) to equation (8), we get

$$\frac{(k+1)!}{k!} Y_1(k+1) = -\sum_{l=0}^{k} \sum_{h_1=k-l}^{N} (1)^l (-1)^{h_1-k+l} \binom{h_1}{k-l} Y_1(l) Y_2(h_1) + \sum_{l=k}^{N} (-10)^{l-k} \binom{l}{k} Y_2(l)$$

$$\frac{(k+1)!}{k!} Y_2(k+1) = \sum_{l=0}^{k} \sum_{h_1=k-l}^{N} (1)^l (-1)^{h_1-k+l} \binom{h_1}{k-l} Y_1(l) Y_2(h_1) - Y_2(k)$$

$$\frac{(k+1)!}{k!} Y_3(k+1) = Y_2(k) - \sum_{l=k}^{N} (-10)^{l-k} \binom{l}{k} Y_2(l)$$

In Figure 2 we can see the effectiveness of differential transformation method applied to the system of differential equations with delay (8).

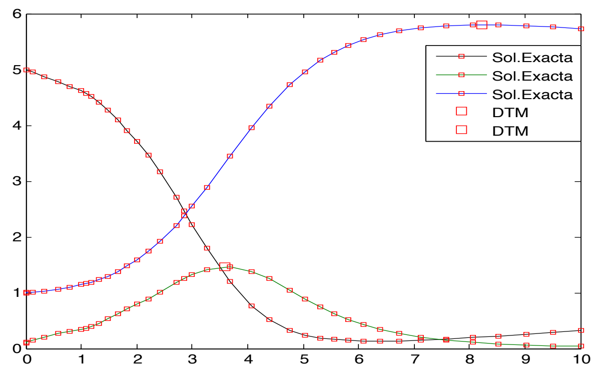

Figure 2: Spread of infection.

6 Conclusions

In this work, we presented the definition and handling of one-dimensional differential transformation method or Zhou's method. Using DTM, delay differential equations were transformed into algebraic equations (iterative equations). The new scheme obtained by using the Zhou's method (to different delay models) yields an analytical solution in the form of a rapidly convergent series. This method makes the solution procedure much more attractive. The figures and tables clearly show the high efficiency of DTM and the convergence of the method for three examples in investigated.

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