

# Pricing American Options Using LU Decomposition

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## Abstract

Numerical solution methods for pricing American options are considered. We propose a second-order accurate Runge-Kutta scheme for the time discretization of the Black-Scholes partial differential equation with an early exercise constraint. We reformulate the algorithm introduced by Brennan and Schwartz into a simple form using LU decomposition and a modified backward substitution with a projection. In addition, we describe a direct solution method given by Elliott and Ockendon and we consider the similarity of these two direct algorithms. Numerical experiments demonstrate that the Runge-Kutta scheme produces smaller errors and less oscillations to numerical solutions than the Crank-Nicolson method. Experiments also show that the Brennan and Schwartz algorithm is much faster than the projected SOR method.

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## 1 Introduction

Financial options are widely used in the field of finance [17]. The valuation of option contracts has been topic of active research during the last decades.

There exist various type of mathematical models for the prices of different kinds of options. For many of these models there are no analytical pricing formulas available and, hence, numerical valuation techniques need to be employed. One of the most famous models for the price of an American option is based on the Black-Scholes partial differential equation [1]. In general, a numerical solution of this option pricing model consists two tasks which are the discretization of the underlying partial differential equation and the solution of complementarity problems.

Surveys of valuation techniques for options are presented for example in [3], [11]. In [3], the speed of computing and the accuracy of computed option prices are compared among many existing valuation methods of American options. For instance, a binomial method, an accelerated binomial method, a trinomial method and an integral equation method are briefly explained and compared in that paper. Explicit and implicit finite difference methods and the binomial method are considered in [11]. The methods are compared numerically in the cases of American call and put options with and without dividends. Furthermore, several option models and solution techniques are studied in the books [17], [28], [29], [31], and references therein.

There are classical methods for the space discretization of partial differential equations and those have been applied in option pricing. For example, a finite difference method is used in [2], [15], [16], [24], a finite element method is considered in [28], and a finite volume method in [10].

The Black-Scholes partial differential equation contains variable coefficients for the first-order and second-order spatial derivatives. The space discretization of this type of partial differential equation may lead to numerical difficulties. The Black-Scholes partial differential equation can be transformed into a constant coefficient diffusion equation and after that the space discretization becomes easier; see [28], [31]. However, there exist several option pricing models for which the above mentioned transformation can not be done and in these cases the discretization of the convection-diffusion type Black-Scholes partial differential equation needs to be performed. The choice of the space discretization scheme for the option pricing problem depends on the form of the partial differential equation. We prefer to use a basic finite difference scheme because of its simplicity and furthermore, because it leads to a tridiagonal space discretization matrix which enables the efficient use of LU decomposition.

The Black-Scholes equation is time-dependent and, thus, numerical time integration methods need to be considered. First-order accurate explicit and implicit Euler schemes are well-known methods for the time discretization [30], [31]. The implicit scheme is only first-order accurate, but it has good stability properties. The Crank-Nicolson method is second-order accurate in time and it is widely used in financial problems [29]. Even though the scheme is unconditionally stable it can produce undesired oscillations to numerical

solutions [10], [21], [26]. One way to improve the stability is to start the time-stepping by a few implicit Euler time steps and then continued with the Crank-Nicolson method. This type of scheme was introduced by Rannacher and it damps undesired oscillations more efficiently than the Crank-Nicolson method does [27]. This scheme is applied for the option pricing problems in [26]. They apply the scheme for the digital call option problem and they showed that the Rannacher time-stepping leads to stable convergence rates while the Crank-Nicolson method does not. The numerical examples were performed with the original nonsmooth payoff function and with a specially smoothed payoff functions.

The second-order backward difference formula and a Runge-Kutta scheme are considered in this paper. They both are second-order accurate time discretizations methods with good stability properties. The Runge-Kutta scheme is L-stable [4]. In numerical experiments, we show that these both schemes lead to more accurate numerical solution than the Crank-Nicolson method. In addition, convergence rates and oscillation problems are studied.

The pricing of an American option is a variational inequality problem [12], and, moreover, it can be formulated as a linear complementarity problem [15], [29]. A basic reference on the complementarity problems is [6]. There are direct and iterative solution algorithms for solving these problems numerically. For example, the widely used projected SOR method is applied for option pricing in [31], a penalty method in [10], and a front-fixing method in [24]. The direct algorithms are applied in [2], [22] and an operator splitting method is used in [18]. The option pricing problem can be also formulated to the form of linear programming. The direct methods and linear programming are considered in [7], [8]. In this paper, we study the direct methods while the projected SOR method is also applied in the numerical experiments.

The rest of this paper is divided into five sections. First, in Section 2, models for the American call and put options are briefly introduced. After that, the space and time discretizations of the Black-Scholes partial differential equation are considered. Two direct algorithms for the complementarity problem resulting from pricing American options are the subject of Section 4. The algorithm by Brennan and Schwartz was introduced in [2] and the algorithm by Elliott and Ockendon [9] is applied for the option pricing in [22]. In that section, we reformulate these algorithms into a simple form using LU decomposition. Moreover, we consider the similarity of these algorithms. The numerical experiments are given in Section 5 and, finally, Section 6 contains conclusions.

## 2 Models for American Options

We begin by introducing mathematical models for American call and put options. The models considered in this paper are based on the Black-Scholes partial differential equation

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 \phi}{\partial x^2} + (r - d)x \frac{\partial \phi}{\partial x} - r\phi = 0, \quad (1)$$

where the function  $\phi(x, t)$  is the price of the option with respect to the underlying asset value  $x$  and time  $t$ . Due to the early exercise possibility of American options an additional constraint

$$\phi(x, t) \geq g(x) \quad (2)$$

has to be introduced in order to avoid arbitrage possibilities. Here,  $g$  is the payoff function of the option contract. The price of the option is obtained by solving the partial differential equation with the previous constraint, boundary conditions, and a final condition [29], [31].

It is well-known [31] that there is a value  $S_f(t)$  for all  $t$  which divides the domain  $(0, \infty)$  into two subdomains  $(0, S_f(t))$  and  $(S_f(t), \infty)$  in such a way that in one of these subdomains the price of the option equals to the payoff function while in the other one it is higher than the payoff. The price of the option satisfies the Black-Scholes equation (1) in the subdomain where it is higher than the payoff. The function  $S_f(t)$  is not known beforehand and it has to be found together with the price of the option. Hence, the option pricing problem is a free boundary problem. The underlying asset value  $S_f(t)$  indicates when the option should be exercised.

Option pricing problems considered in this paper are posed in an infinite region  $[0, \infty) \times [0, T]$  with Dirichlet boundary conditions and a final condition. In order to solve these problems numerically, we reformulate problems in a truncated region  $[0, X] \times [0, T]$ . The truncation point  $X$  has to be sufficiently far in order to avoid excessive error due to the truncation. On the other hand unnecessarily large value of  $X$  increases computational cost. The choice of  $X$  is considered in [20], for example. Furthermore, we transform final value problems into a more familiar form of initial value problems.

### 2.1 American call option

An American call option gives a right to buy the underlying asset for the exercise price  $E$  any time before expiry. The value of the option is obtained

by solving the linear complementarity problem

$$\left\{ \begin{array}{l} \frac{\partial \phi}{\partial t} - \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 \phi}{\partial x^2} - (r - d)x \frac{\partial \phi}{\partial x} + r\phi \geq 0, \\ \phi(x, t) \geq g(x), \\ \left( \frac{\partial \phi}{\partial t} - \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 \phi}{\partial x^2} - (r - d)x \frac{\partial \phi}{\partial x} + r\phi \right) (\phi - g) = 0, \end{array} \right. \quad (3)$$

where  $(x, t) \in [0, X] \times [0, T]$ , with the boundary conditions

$$\phi(0, t) = 0 \quad \text{and} \quad \phi(X, t) = X - E, \quad (4)$$

and with the initial value  $\phi(x, 0) = g(x)$ . Here,  $g$  is the payoff function

$$g(x) = \max(x - E, 0). \quad (5)$$

We denote the risk free interest rate by  $r$ , the volatility by  $\sigma$ , and the constant dividend yield by  $d$ . In the domain  $(0, S_f(t))$  at time  $t$ , the solution satisfies the Black-Scholes partial differential equation and the solution is equal to the payoff function in the domain  $(S_f(t), X)$ .

## 2.2 American put option

An American put option gives a right to sell the underlying asset for the exercise price  $E$  any time before expiry. The initial value problem for the American put option is

$$\left\{ \begin{array}{l} \frac{\partial \phi}{\partial t} - \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 \phi}{\partial x^2} - (r - d)x \frac{\partial \phi}{\partial x} + r\phi \geq 0, \\ \phi(x, t) \geq g(x), \\ \left( \frac{\partial \phi}{\partial t} - \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 \phi}{\partial x^2} - (r - d)x \frac{\partial \phi}{\partial x} + r\phi \right) (\phi - g) = 0, \end{array} \right. \quad (6)$$

where  $(x, t) \in [0, X] \times [0, T]$ , with the boundary conditions

$$\phi(0, t) = E \quad \text{and} \quad \phi(X, t) = 0, \quad (7)$$

and with the initial value  $\phi(x, 0) = g(x)$ . The payoff function for the put option contract is

$$g(x) = \max(E - x, 0). \quad (8)$$

The solution  $\phi$  is equal to the payoff function at time  $t$  in the domain  $(0, S_f(t))$ , while in the domain  $(S_f(t), X)$ , it satisfies the Black-Scholes partial differential equation.

### 3 Discretization of the Black-Scholes equation

In the following, we consider the discretization of the Black-Scholes partial differential equation

$$\frac{\partial \phi}{\partial t} - \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 \phi}{\partial x^2} - (r - d)x \frac{\partial \phi}{\partial x} + r\phi = 0. \quad (9)$$

We apply a uniform grid in the computational domain  $[0, X] \times [0, T]$  which is formed with a space step  $\Delta x = X/(m + 1)$  and a time step  $\Delta t = T/n$ . Moreover, we use the notation

$$\phi_i^{(j)} \approx \phi(x_i, t_j) = \phi(i\Delta x, j\Delta t), \quad (10)$$

where  $i = 0, \dots, m + 1$  and  $j = 0, \dots, n$ , for the numerical approximation of the solution. The efficiency of numerical solution can be improved by coordinate transformations or using nonuniform grids [5], [29]. For simplicity we do not consider such approaches here, but the proposed methods can be readily extended for these techniques.

#### 3.1 Space discretization

The second-order accurate central finite differences

$$\frac{\partial \phi}{\partial x}(x_i, t) \approx \frac{\phi_{i+1}(t) - \phi_{i-1}(t)}{2\Delta x} \quad \text{and} \quad \frac{\partial^2 \phi}{\partial x^2}(x_i, t) \approx \frac{\phi_{i+1}(t) - 2\phi_i(t) + \phi_{i-1}(t)}{(\Delta x)^2} \quad (11)$$

are used for the approximation of the space derivatives of the Black-Scholes partial differential equation [23], [29]. These lead to the semi-discrete equation

$$\frac{\partial \phi}{\partial t} - \frac{1}{2} \left( \sigma^2 i^2 - (r - d)i \right) \phi_{i-1} + \left( \sigma^2 i^2 + r \right) \phi_i - \frac{1}{2} \left( \sigma^2 i^2 + (r - d)i \right) \phi_{i+1} = 0, \quad (12)$$

where  $i = 1, \dots, m$ . This equation can be expressed more compactly in the form

$$\frac{\partial \phi}{\partial t} + \mathbf{B}\phi = 0, \quad (13)$$

where the matrix  $\mathbf{B}$  is tridiagonal. With this discretization  $\mathbf{B}$  has an  $M$ -matrix property if  $\sigma^2 > r - d$ . This property guarantees that the space discretization does not cause undesired oscillations into the numerical solution [32], [33].

### 3.2 Time discretization

Next, the time discretization of the semi-discrete equation (13) is discussed. We consider the Crank-Nicolson scheme, the second-order backward difference formula (BDF2), and a Runge-Kutta scheme. The stability of time discretization schemes is considered in [4], [21], for example. It is demonstrated that for some problems the Crank-Nicolson method performs poorly while L-stable methods are more accurate.

The Crank-Nicolson time discretization scheme can be interpreted to be the average of the implicit and explicit Euler schemes. The scheme

$$\frac{1}{\Delta t}(\phi^{(j+1)} - \phi^{(j)}) + \frac{1}{2}\mathbf{B}(\phi^{(j+1)} + \phi^{(j)}) = 0, \quad (14)$$

is second-order accurate and unconditionally stable [29]. Despite this stability property, it has been noticed, for example in [14], [21], [26], that a nonsmooth initial value can lead to numerical solutions with oscillations. One possible way to prevent the oscillation problem is to start the time-stepping by a few implicit Euler steps and then continue with the Crank-Nicolson scheme [26], [27].

The BDF2 formula is more stable than the Crank-Nicolson scheme [14], [25], and due to this high frequency oscillations are sufficiently damped out in time. This second-order accurate scheme is

$$\frac{1}{\Delta t}\left(-\frac{4}{3}\phi^{(j)} + \phi^{(j+1)} + \frac{1}{3}\phi^{(j-1)}\right) + \frac{2}{3}\mathbf{B}\phi^{(j+1)} = 0, \quad (15)$$

where  $\phi^{(j+1)}$  is computed by using data from time steps  $j$  and  $j - 1$ . The numerical integration has to be started with some other discretization method because this scheme needs data from two previous time steps. For example, we compute the vector  $\phi^{(1)}$  using the implicit Euler scheme.

The third time discretization which we apply in numerical experiments is an L-stable Runge-Kutta scheme [4]. This two-step method has the form

$$\begin{cases} (\mathbf{I} + \theta\Delta t\mathbf{B})\bar{\phi}^{(j+1)} = (\mathbf{I} - (1 - \theta)\Delta t\mathbf{B})\phi^{(j)}, \\ (\mathbf{I} + \theta\Delta t\mathbf{B})\phi^{(j+1)} = (\mathbf{I} - \frac{1}{2}\Delta t\mathbf{B})\phi^{(j)} - (\frac{1}{2} - \theta)\Delta t\mathbf{B}\bar{\phi}^{(j+1)}, \end{cases} \quad (16)$$

where the parameter is chosen to be

$$\theta = 1 - 1/\sqrt{2}. \quad (17)$$

This scheme performs two fractional steps and, hence, two systems of linear equations need to be solved at each time step while the Crank-Nicolson method and the BDF2 formula requires only one solution. In addition, it is worth



## 4.1 Brennan and Schwartz algorithm with LU decomposition

Brennan and Schwartz introduced an algorithm for pricing American put options in [2]. The solution algorithm is based on a Gaussian elimination where the early exercise constraint of the option contract is handled in a simple manner. This algorithm is analysed in [19]. In this section, we reformulate this algorithm in such a way that it can be used for the American call option problem (3) and, furthermore, we give the algorithm in a form where LU decomposition of the discretization matrix can be used.

The space and time discretization of the complementarity problems (3) and (6) lead to a sequence of stationary complementarity problems. Hence, at each time step  $j$ ,  $j = 0, \dots, n - 1$ , the problem

$$\begin{cases} \mathbf{A}\phi^{(j+1)} \geq \mathbf{b}^{(j)}, \\ \phi^{(j+1)} \geq \mathbf{g}, \\ (\mathbf{A}\phi^{(j+1)} - \mathbf{b}^{(j)})^T (\phi^{(j+1)} - \mathbf{g}) = 0, \end{cases} \quad (21)$$

needs to be solved. Here, the matrix  $\mathbf{A}$  results from the discretization of the partial differential equation and the vector  $\mathbf{g}$  contains the grid point values of the payoff function  $g$ . In the following, we consider how the Brennan and Schwartz algorithm is applied at every time step to solve the problem (21). As we observe, once LU decomposition has been made, it suffices only to make a forward and a modified backward substitution at each time step.

First, we describe the Brennan and Schwartz algorithm briefly for the American call option. The algorithm is the following: The Gaussian elimination transforms each row of the system of linear equations  $\mathbf{A}\phi = \mathbf{b}$  to be the form

$$p_i\phi_i + s_i\phi_{i+1} = k_i, \quad (22)$$

for  $i = 1, \dots, m$ , where the coefficients can be chosen to be

$$\begin{aligned} p_1 &= a_{11}, & s_1 &= a_{12}, & k_1 &= b_1, \\ p_i &= a_{ii} - \frac{a_{ii-1}}{p_{i-1}} a_{i-1i}, & s_i &= a_{ii+1}, & k_i &= b_i - \frac{a_{ii-1}}{p_{i-1}} k_{i-1}, \\ p_m &= a_{mm} - \frac{a_{mm-1}}{p_{m-1}} a_{m-1m} & \text{and} & & k_m &= b_m - \frac{p_{m-1}}{a_{mm-1}} k_{m-1}. \end{aligned}$$

These coefficients are formed starting from the first row of the system of linear equations while in the original algorithm the elimination was started from

the last row. The reason is that we consider call options and the article [2] considers put options. The solution  $\phi$  for the problem (21) is obtained by using the equations (22) and the payoff function of the call option. The algorithm starts by solving  $\phi_m$  from the equation (22) when  $i = m$ ; if  $\phi_m$  is less than the payoff value then set  $\phi_m = g_m$ . Then proceed to solve  $\phi_{m-1}$ , etc. For details, see the original article [2].

The Brennan and Schwartz algorithm can be reformulated to the form where LU decomposition is applied. In a basic form, the solution of the system of linear equations using LU decomposition consists of forward and backward substitutions:

$$\mathbf{L}\mathbf{y} = \mathbf{b} \quad \text{and} \quad \mathbf{U}\mathbf{x} = \mathbf{y}, \quad (23)$$

where  $\mathbf{L}$  is a lower triangular matrix and  $\mathbf{U}$  is an upper triangular matrix. Next, we give the Brennan and Schwartz algorithm in a form where only the previous backward substitution is modified.

In the following LU decomposition, the lower and the upper triangular matrices are formed in such a way that elements of  $\mathbf{U}$  corresponds to  $p_i$  and  $s_i$  in the Brennan and Schwartz algorithm. The similarity can be seen by noticing:

$$l_{ii-1} = \frac{a_{ii-1}}{p_{i-1}}, \quad l_{ii} = 1, \quad u_{ii} = p_i, \quad u_{ii+1} = s_i, \quad \text{and} \quad y_i = k_i. \quad (24)$$

Reformulation for the Brennan and Schwartz algorithm using LU decomposition of the discretization matrix reads

**Algorithm 1: Brennan and Schwartz algorithm with LU decomposition**

```

u11 = a11
u12 = a12
do i = 2, ..., m - 1
    lii-1 = aii-1/ui-1i-1
    uii = aii - lii-1 ui-1i
    uii+1 = aii+1
end do
lmm-1 = amm-1/um-1m-1
umm = amm - lmm-1 um-1m

y1 = b1
do i = 2, ..., m
    yi = bi - lii-1 yi-1
end do

```

```

 $\phi_m = y_m / u_{mm}$ 
 $\phi_m = \max(\phi_m, g_m)$ 
do  $i = m - 1, \dots, 1$ 
     $\phi_i = (y_i - u_{i+1} \phi_{i+1}) / u_{ii}$ 
     $\phi_i = \max(\phi_i, g_i)$ 
end do

```

It is obvious that the use of the max-function in the backward substitution is possible because of the form of the solution of the option pricing problem. This algorithm is for the time step  $j$  of the problem (21) and so, the right-hand side vector  $\mathbf{b}^{(j)}$  needs to be updated at each time step.

## 4.2 Brennan and Schwartz algorithm for a transformed problem

In order to compare the Brennan and Schwartz algorithm with another direct algorithm, we transform the call option problem to the form where the early exercise constraint equals to zero function instead of the payoff function. This transformation is made by using the following substitutions for the problem (21):

$$\mathbf{z}^{(j+1)} = \boldsymbol{\phi}^{(j+1)} - \mathbf{g} \quad \text{and} \quad \mathbf{v}^{(j)} = \mathbf{b}^{(j)} - \mathbf{A}\mathbf{g}. \quad (25)$$

With these notations, the complementarity problems can be reformulated into a form:

$$\left\{ \begin{array}{l} \mathbf{A}\mathbf{z}^{(j+1)} \geq \mathbf{v}^{(j)}, \\ \mathbf{z}^{(j+1)} \geq 0, \\ (\mathbf{A}\mathbf{z}^{(j+1)} - \mathbf{v}^{(j)})^T \mathbf{z}^{(j+1)} = 0, \end{array} \right. \quad (26)$$

where  $j = 0, \dots, n - 1$ .

In the following, we formulate the Brennan and Schwartz algorithm using another LU decomposition which has ones on the diagonal of  $\mathbf{U}$ . Again a projection is added into the backward substitution. The Brennan and Schwartz algorithm for the transformed problem (26) with the alternative LU decomposition reads

**Algorithm 2: Brennan and Schwartz algorithm for the transformed problem**

```

 $\bar{l}_{11} = a_{11}$ 
 $\bar{u}_{12} = a_{12} / \bar{l}_{11}$ 
do  $i = 2, \dots, m - 1$ 

```

$$\begin{aligned} \bar{l}_{ii-1} &= a_{ii-1} \\ \bar{l}_{ii} &= a_{ii} - \bar{l}_{ii-1} \bar{u}_{i-1i} \\ \bar{u}_{ii+1} &= a_{ii+1} / \bar{l}_{ii} \end{aligned}$$

**end do**

$$\begin{aligned} \bar{l}_{mm-1} &= a_{mm-1} \\ \bar{l}_{mm} &= a_{mm} - \bar{l}_{mm-1} \bar{u}_{m-1m} \end{aligned}$$

$$\bar{y}_1 = v_1 / \bar{l}_{11}$$

**do**  $i = 2, \dots, m$

$$\bar{y}_i = (v_i - \bar{l}_{ii-1} \bar{y}_{i-1}) / \bar{l}_{ii}$$

**end do**

$$z_m = \bar{y}_m$$

$$z_m = \max(z_m, 0)$$

**do**  $i = m - 1, \dots, 1$

$$z_i = \bar{y}_i - \bar{u}_{ii+1} z_{i+1}$$

$$z_i = \max(z_i, 0)$$

**end do**

The solution for the original problem (21) is obtained by  $\phi^{(j+1)} = \mathbf{z}^{(j+1)} + \mathbf{g}$ .

### 4.3 Elliott-Ockendon algorithm with LU decomposition

The second algorithm which we consider was presented by Elliott and Ockendon. They gave the algorithm for the linear complementarity problem in [9] and this direct algorithm is applied for the American option pricing problems in [22]. According to [15], for example, the complementarity problems can be expressed in the form:

$$\mathbf{w}^{(j+1)} = \mathbf{A}\mathbf{z}^{(j+1)} - \mathbf{v}^{(j)}, \quad \mathbf{w}^{(j+1)} \geq 0, \quad \mathbf{z}^{(j+1)} \geq 0 \quad \text{and} \quad (\mathbf{w}^{(j+1)})^T \mathbf{z}^{(j+1)} = 0, \quad (27)$$

where  $j = 0, \dots, n - 1$ . This form of the complementarity problem is also considered in [9] and [22]. They assume that the discretization matrix  $\mathbf{A}$  is an  $M$ -matrix. Furthermore, it is assumed that the vector  $\mathbf{v}^{(j)}$  has an specified form; see details in [9]. In the following, the solution method is again considered only at one time step, and so the time step index  $j$  is omitted.

The Elliott-Ockendon algorithm in the reference [9] reads

#### Algorithm 3: Elliott-Ockendon algorithm

**decompose**       $i = 1, \quad \alpha_1 = a_{11}, \quad f_1 = v_1/a_{11}$

```

step:       $i \rightarrow i + 1$ 
               $\gamma_{i-1} = a_{i-1i}/\alpha_{i-1}$ 
               $\alpha_i = a_{ii} - a_{ii-1} \gamma_{i-1}$ 
               $f_i = (v_i - a_{ii-1} f_{i-1})/\alpha_i$ 

if  $f_i > v_{i+1}/a_{i+1i}$  then go to step

backsolve   $k = i$ 
               $z_k = f_k$ 
               $i = k - 1$   $(-1)$   $1$ 
               $z_i = f_i - \gamma_i z_{i+1}$ 

```

Parts of this algorithm are similar to LU decomposition. That is,  $\gamma_i$  and  $\alpha_i$  are elements of LU decomposition and  $f_i$  is the intermediate vector after the forward substitution. In the Elliott-Ockendon algorithm, only those elements are computed which are needed.

Next, we reformulate this Elliott-Ockendon algorithm in such a way that LU decomposition is formed fully and the early exercise constraint is taken into account in a modified backward substitution. The elements of this LU decomposition correspond to those of the Elliott-Ockendon algorithm in the following way:

$$\bar{l}_{ii} = \alpha_i, \quad \bar{u}_{ii+1} = \gamma_i, \quad \text{and} \quad \bar{y}_i = f_i. \quad (28)$$

Taking advantage of LU decomposition, the Elliott-Ockendon algorithm has the form:

**Algorithm 4: Elliott-Ockendon algorithm with LU decomposition**

```

 $\bar{l}_{11} = a_{11}$ 
 $\bar{u}_{12} = a_{12}/\bar{l}_{11}$ 
do  $i = 2, \dots, m - 1$ 
     $\bar{l}_{ii-1} = a_{ii-1}$ 
     $\bar{l}_{ii} = a_{ii} - \bar{l}_{ii-1} \bar{u}_{i-1i}$ 
     $\bar{u}_{ii+1} = a_{ii+1}/\bar{l}_{ii}$ 
end do
 $\bar{l}_{mm-1} = a_{mm-1}$ 
 $\bar{l}_{mm} = a_{mm} - \bar{l}_{mm-1} \bar{u}_{m-1m}$ 

 $\bar{y}_1 = v_1/\bar{l}_{11}$ 
 $i = 1$ 
do while  $(\bar{y}_i > v_{i+1}/a_{i+1i})$ 

```

```

       $i = i + 1$ 
       $\bar{y}_i = (v_i - a_{ii-1} \bar{y}_{i-1}) / \bar{l}_{ii}$ 
end do

 $k = i$ 
do  $i = m, \dots, k + 1$ 
       $z_i = 0$ 
end do

 $z_k = \bar{y}_k$ 
do  $i = k - 1, \dots, 1$ 
       $z_i = \bar{y}_i - \bar{u}_{ii+1} z_{i+1}$ 
end do

```

Again, this algorithm should be applied at every time step and the solution for the original option pricing problem is obtained as  $\phi^{(j+1)} = \mathbf{z}^{(j+1)} + \mathbf{g}$ .

#### 4.4 Comparison of two direct algorithms

At each time step the solution of the linear complementarity problem satisfies the Black-Scholes partial differential equation in one region and in the other region it equals the early exercise constraint. Thus, at the time step  $j$  there is one discretization node  $k_j$  which divides the  $x$ -axis for these two separate regions. Next, we study how this node  $k_j$  can be found during the forward substitution. We show that the solution satisfies the partial differential equation at the nodes where the intermediate value after the forward substitution is positive and the solution equals to the early exercise constraint (here zero function) at the nodes where the intermediate value after the forward substitution is negative. This kind of condition can be found by studying both direct algorithms and this is what is done in the following.

As mentioned earlier, we assume that the solution has a specified form. More precisely, it is assumed that the solution  $\mathbf{z}^{(j)}$  has the form

$$z_i^{(j)} > 0, \quad i = 0, \dots, k_j, \quad \text{and} \quad z_i^{(j)} = 0, \quad i = k_j + 1, \dots, m + 1, \quad (29)$$

at each time step  $j$ .

First, we study the Elliott-Ockendon algorithm and especially Algorithm 4. The condition in the **do while** loop can be reformulate using the following equivalence relations:

$$\bar{y}_{i+1} = (v_{i+1} - a_{i+1i} \bar{y}_i) / \bar{l}_{ii} > 0$$

$$\iff v_{i+1} - a_{i+1i} \bar{y}_i > 0$$

$$\iff v_{i+1} > a_{i+1i} \bar{y}_i$$

$$\iff \bar{y}_i > v_{i+1}/a_{i+1i},$$

where  $a_{i+1i}$  is negative due to the  $M$ -matrix property. Also the positivity of  $\bar{l}_{ii}$  follows easily from this property. We assume that  $k_j$  is the smallest index for which it holds that  $\bar{y}_{k_j+1} < 0$ . According to the Elliott-Ockendon algorithm and equivalences above, we notice that  $z_{i+1} = 0$  for  $i = k_j, \dots, m$  and, furthermore, the rest of the solution can be computed using the backward substitution which is  $z_i = \bar{y}_i - \bar{u}_{ii+1} z_{i+1}$  for  $i = k_j, \dots, 1$ .

This leads to the conclusion that also the Elliott-Ockendon algorithm can be reformulated into the form where the forward substitution is carried out for all components and the early exercise constraint is taken into account in the modified backward substitution. In fact, this modified backward substitution is similar to the last loop of Algorithm 2.

Similarly, in the Brennan and Schwartz algorithm the node  $k_j$  can be found by studying the forward substitution. First, the last loop of Algorithm 2 can be written in the form:

```

do while ( $\bar{y}_i - \bar{u}_{ii+1} z_{i+1} < 0$ )
     $z_i = 0$ 
     $i = i - 1$ 
end do

 $k = i$ 
do  $i = k, 1$ 
     $z_i = \bar{y}_i - \bar{u}_{ii+1} z_{i+1}$ 
end do

```

From this and the assumption on the specific form of the solution, it follows that the solution  $z_i$  is zero when  $\bar{y}_i$  is negative. Hence, also in this Brennan and Schwartz algorithm we can find the node  $k_j$  by studying the sign of  $\bar{y}_i$  in the forward substitution.

## 5 Numerical Experiments

In this section, we present numerical experiments with the methods discussed in the previous sections. We show that in the option pricing the BDF2 formula and the Runge-Kutta scheme are more efficient time discretizations than

the widely used Crank-Nicolson method. Furthermore, we compare required CPU times between the Brennan and Schwartz algorithm and the projected SOR method with different values of  $m$  and  $n$ . Finally, we compare the delta functions of the option prices computed using the Crank-Nicolson method and the Runge-Kutta method.

In order to report maximum errors for numerical solutions, we computed reference solutions for the American call and put option problems numerically. These reference solutions were computed with a very fine grid defined by  $(m, n) = (131072, 4096)$ , the central finite difference scheme, the Runge-Kutta scheme, and the Brennan and Schwartz algorithm. In the following examples, the computational domain is  $[0, 50] \times [0, 1]$ . Numerical experiments were performed on an HP J5600 workstation.

## 5.1 Example I

First, we compare the convergence rates of the implicit Euler scheme, the Crank-Nicolson method, the BDF2 formula and the Runge-Kutta scheme. Also the stability of these schemes is studied numerically. The spatial derivatives were approximated by the central finite differences described in Section 3.1 and the complementarity problems at each time step were solved with the Brennan and Schwartz algorithm. In this numerical experiment, the American call option problem (3) is described by the parameter values  $\sigma = 0.6$ ,  $r = 0.25$ ,  $E = 10$ ,  $d = 0.2$  and  $T = 1$ .

For parabolic partial differential equations the Crank-Nicolson method, the BDF2 formula and the Runge-Kutta scheme are second-order accurate in time. Due to the nonsmooth initial function, the convergence rates of these three methods are reduced. The BDF2 formula and Runge-Kutta scheme are more accurate than only unconditionally stable Crank-Nicolson method. This can be seen from Tables 1 and 2 where we report the maximum errors of numerical solutions. We have also computed ratios between the error with  $n$  and the error with  $n/2$ . This estimates the convergence rate of the time discretization schemes. With moderate time steps the convergence rate and accuracy of the Runge-Kutta scheme are higher than with other schemes. Especially the convergence of the Crank-Nicolson method is slow with larger time steps. The first-order accurate implicit Euler scheme converges as expected. It is never as accurate as the BDF2 formula and the Runge-Kutta scheme, but for large time steps it is more accurate than the Crank-Nicolson method.

In the case of the BDF2 formula and the Runge-Kutta scheme, the numerical solutions become more accurate when the number of space step  $m$  is increased from 2047 to 8191; see Tables 1 and 2. While this is not true for the Crank-Nicolson method. Due to the poor stability properties of the Crank-Nicolson method the accuracy of the solution decreases when the number of

$n$	Implicit-Euler		Crank-Nicolson		BDF2		Runge-Kutta	
	error	ratio	error	ratio	error	ratio	error	ratio
2	1.905E-1		2.739E-1		1.111E-1		2.307E-2	
4	1.015E-1	1.88	1.403E-1	1.95	3.084E-2	3.60	3.680E-3	6.27
8	5.305E-2	1.91	6.929E-2	2.02	7.136E-3	4.32	7.892E-4	4.66
16	2.732E-2	1.94	3.265E-2	2.12	2.056E-3	3.47	1.830E-4	4.31
32	1.392E-2	1.96	1.422E-2	2.30	6.381E-4	3.22	5.099E-5	3.59
64	7.044E-3	1.98	5.237E-3	2.71	1.944E-4	3.28	1.606E-5	3.17
128	3.547E-3	1.99	1.271E-3	4.12	5.602E-5	3.47	3.776E-6	4.25
256	1.780E-3	1.99	9.369E-5	13.57	1.524E-5	3.68	2.105E-6	1.79
512	8.920E-4	2.00	2.182E-6	42.93	4.100E-6	3.72	2.105E-6	1.00

Table 1: The maximum errors and the time convergence rates when  $m = 2047$ .

space steps is increased. In Figure 1, we have plotted error functions in the case of the Crank-Nicolson method and the Runge-Kutta scheme for various number of time steps,  $n = 8, 16, 32$ , and for a fixed number of space step,  $m = 2047$ . Undesired oscillations are clearly visible in the errors for the Crank-Nicolson method and the errors are significantly smaller for the Runge-Kutta scheme.

$n$	Implicit-Euler		Crank-Nicolson		BDF2		Runge-Kutta	
	error	ratio	error	ratio	error	ratio	error	ratio
2	1.905E-1		2.773E-1		1.111E-1		2.307E-2	
4	1.016E-1	1.88	1.438E-1	1.93	3.084E-2	3.60	3.681E-3	6.27
8	5.305E-2	1.91	7.280E-2	1.97	7.136E-3	4.32	7.904E-4	4.66
16	2.732E-2	1.94	3.610E-2	2.02	2.056E-3	3.47	1.841E-4	4.29
32	1.392E-2	1.96	1.751E-2	2.06	6.382E-4	3.22	5.107E-5	3.60
64	7.044E-3	1.98	8.184E-3	2.14	1.944E-4	3.28	1.473E-5	3.47
128	3.547E-3	1.99	3.551E-3	2.30	5.600E-5	3.47	3.920E-6	3.76
256	1.780E-3	1.99	1.306E-3	2.72	1.519E-5	3.69	9.892E-7	3.96
512	8.919E-4	2.00	3.161E-4	4.13	3.966E-6	3.83	2.382E-7	4.15

Table 2: The maximum errors and the time convergence rates when  $m = 8191$ .

## 5.2 Example II

Next, we compare required CPU times when the Brennan and Schwartz algorithm and the projected SOR method are applied for the American call option problem. The discretization was performed with the central finite difference schemes and the Runge-Kutta scheme. Furthermore, the same parameter values were used as in the first numerical experiment.

The stopping criterion for the iterative projected SOR method was chosen in such a way that the maximum error was at the most 10 percent larger

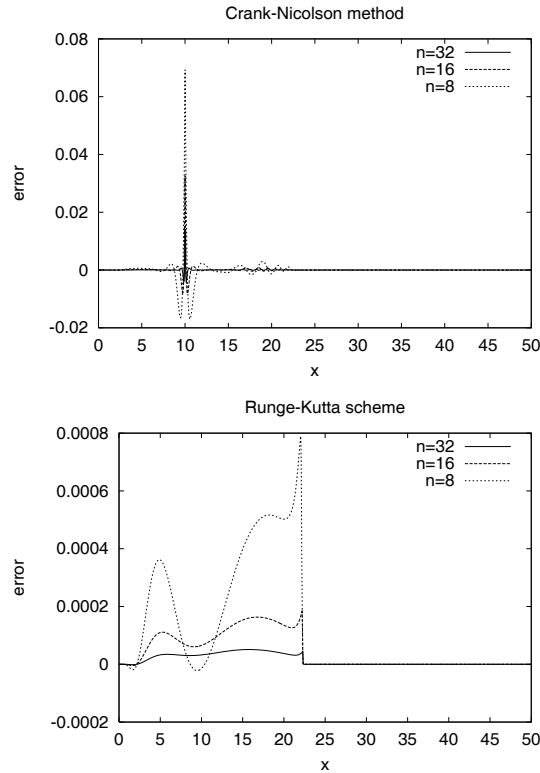


Figure 1: The error functions of the numerical solutions in the cases of the Crank-Nicolson method (left) and the Runge-Kutta scheme (right) when  $m = 2047$  and  $n$  varies.

than with the direct Brennan and Schwartz algorithm. The overrelaxation parameter  $\omega$  in the projected SOR was optimized so that iterations would converge as fast as possible. The values of  $\omega$  are given for the different values of  $m$  and  $n$  in Table 3.

The results of this example are reported in Table 3. For the Brennan and Schwartz algorithm the required CPU time increases linearly with respect to the number of unknowns. The CPU time for every  $(m, n)$  pair are several times smaller than the corresponding time for the projected SOR method. The CPU times for the projected SOR method grow approximately like  $(mn)^{1.35}$ .

### 5.3 Example III

In the last numerical experiment, we study the influence of the time discretization schemes to the delta function, which is the derivative of the option price with respect to  $x$ . In Figure 2, we have plotted the delta functions for the American call and put options in the cases of the Crank-Nicolson method and the Runge-Kutta scheme. The Runge-Kutta scheme leads to smooth

$m$	$n$	PSOR				B-S	
		error	CPU	$\omega$	iter ave	error	CPU
127	16	2.684E-4	0.026	1.5	14.063	2.641E-4	0.006
255	32	1.393E-4	0.146	1.6	20.219	1.341E-4	0.022
511	64	2.271E-5	1.191	1.71	44.047	2.144E-5	0.089
1023	128	1.221E-5	6.235	1.79	55.133	1.190E-5	0.353
2047	256	2.182E-6	35.230	1.88	83.223	2.105E-6	1.392
4095	512	5.705E-7	311.672	1.90	184.586	5.765E-7	5.586

Table 3: The maximum errors and the CPU times in the cases of the projected SOR method and the Brennan and Schwartz (B-S) algorithm.

delta functions while undesired oscillations arising from the use of the Crank-Nicolson method can be clearly seen. This oscillation is focused near to the exercise price  $E$ , which is where the space derivative of the initial function is discontinuous. The parameter values were same as in the first experiment and the grid defined by  $(m, n) = (2047, 32)$  was used. In order to show oscillations clearly, the number of time steps  $n$  is relatively small.

## 6 Conclusions

In this paper, we have considered finite difference methods for pricing American options. We have studied time discretization schemes for the Black-Scholes partial differential equation and introduced a reformulation of the Brennan and Schwartz algorithm.

We applied the BDF2 formula and the Runge-Kutta scheme for the time discretization and in the numerical experiments these were found out to be more accurate than the Crank-Nicolson method. The discontinuous first derivative of the payoff function decreases convergence rate. We demonstrated that the convergence rates of the BDF2 formula and Runge-Kutta scheme are better than the convergence rate of the Crank-Nicolson method. In addition, we showed that the L-stable Runge-Kutta scheme produces much less oscillations than only unconditionally stable Crank-Nicolson method.

In the option pricing, the complementarity problems need to be solved at each time step. We reformulated the Brennan and Schwartz algorithm in a way that LU decomposition can be used. Furthermore, we studied the Elliott-Ockendon algorithm and we showed that also this algorithm can be formulated into a form where LU decomposition is employed. In these reformulations, the early exercise constraint is handled in a modified backward substitution using a projection.

Furthermore, in the numerical experiments we compared the required CPU

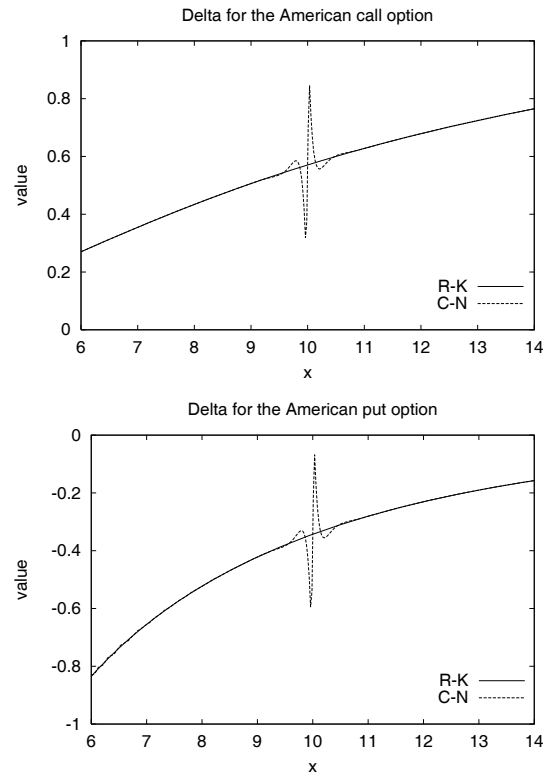


Figure 2: The delta functions in the cases of the American call option (left) and put option (right) when the Crank-Nicolson method and the Runge-Kutta scheme are applied,  $(m, n) = (2047, 32)$ .

times between the direct Brennan and Schwartz algorithm and the iterative projected SOR method. It was observed that the Brennan and Schwartz algorithm was several times faster on coarser grids and tens of times faster on finer grids. Besides, it is difficult to find an optimal stopping criterion and overrelaxation parameter for the projected SOR method while the Brennan and Schwartz algorithm is parameter free and, thus, much easier to use.

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