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Numerical Blow-up Solutions of Localized Semilinear Parabolic Equations

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Abstract

This paper concerns the study of the numerical approximation for the following initial-boundary value problem:

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + (u(0,t))^p, \quad (x,t) \in (-l,l) \times (0,T), \\ u(-l,t) = 0, \quad u(l,t) = 0, \quad t \in (0,T), \\ u(x,0) = u_0(x) \geq 0, \quad x \in (-l,l), \end{cases}$$

where $p > 1$, $l = \frac{1}{2}$. Under some assumptions, we prove that the solution of a semidiscrete form of the above problem blows up in a finite time and estimate its semidiscrete blow-up time. We also show that the semidiscrete blow-up time in certain cases converges to the real one when the mesh size tends to zero. Finally, we give some numerical experiments to illustrate our analysis.

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1 Introduction

We consider the following initial-boundary value problem:

$$u_t(x,t) = u_{xx}(x,t) + (u(0,t))^p, \quad (x,t) \in (-l,l) \times (0,T), \tag{1}$$

$$u(-l,t)=0, \quad u(l,t)=0, \quad t\in(0,T), \tag{2}$$

$$u(x,0)=u_0(x)\geq 0, \quad x\in(-l,l), \tag{3}$$

where $p>1$, $l=\frac{1}{2}$, $u_0(x)$ is a function which is bounded and symmetric. In addition, $u_0(x)$ is nondecreasing on the interval $(-l,0)$ and $u_0^{''}(x)+u^p(0,t)\geq 0$ on $(-l,l)$. Here $(0,T)$ is the maximal time interval of existence of the solution u. The time T may be finite or infinite. When T is infinite, we say that the solution u exists globally. When T is finite, then the solution u develops a singularity in a finite time, namely

$$\lim_{t\to T}\|u(x,t)\|_\infty=+\infty,$$

where $\|u(x,t)\|_\infty=\max_{0\leq x\leq 1}|u(x,t)|$. In this case, we say that the solution u blows up in a finite time and the time T is called the blow-up time of the solution u.

The above problem represents a model in physical phenomena where the reaction is driven by the temperature at a single site. This kind of phenomena is observed in biological systems and in chemical reaction diffusion processes in which the reaction takes place only at same local sites. For more physical motivation see for instance [4] and [7].

In this paper, we are interesting in the numerical study of the above problem. Let I be a positive integer, and consider the grid $x_i=ih$, $0\leq i\leq I$, where $h=2l/I$. We approximate the solution u of (1)–(3) by the solution $U_h(t)=(U_0(t),U_1(t),\dots,U_I(t))^T$ of the following semidiscrete equations

$$\frac{d}{dt}U_i(t)=\delta^2U_i(t)+U_k^p(t), \quad 1\leq i\leq I-1, \quad t\in(0,T_b^h), \tag{4}$$

$$U_0(t)=0, \quad U_I(t)=0, \quad t\in(0,T_b^h), \tag{5}$$

$$U_i(0)=\varphi_i\geq 0, \quad 0\leq i\leq I, \tag{6}$$

where k is the integer part of the number $I/2$,

$$\delta^2U_i(t)=\frac{U_{i+1}(t)-2U_i(t)+U_{i-1}(t)}{h^2}, \quad 1\leq i\leq I-1,$$

$$\varphi_0=0, \quad \varphi_I=0, \quad \varphi_i=\varphi_{I-i}, \quad 0\leq i\leq I, \quad \delta^+\varphi_i>0, \quad 0\leq i\leq k-1,$$

$$\delta^+\varphi_i=\frac{\varphi_{i+1}-\varphi_i}{h}.$$

Here $(0, T_b^h)$ is the maximal time interval on which $\|U_h(t)\|_\infty < +\infty$ with $\|U_h(t)\|_\infty = \max_{0\leq i\leq I} |U_i(t)|$. When the time T_b^h is finite, we say that $U_h(t)$ blows up in a finite time and the time T_b^h is called the blow-up time of the solution $U_h(t)$.

The theoretical study of blow-up solutions for semilinear parabolic equations has been the subject of investigations of many authors (see [3],[5], [8], [9] and the references cited therein). In particular in [5], [8] and [9], the authors have proved that under some assumptions, the solution of (1)–(3) blows up in a finite time and the blow-up time is estimated. Here we are interesting in the numerical study using a semidiscrete form of (1)–(3). We give some assumptions under which the solution of (4)–(6) blows up in a finite time and estimate its semidiscrete blow-up time. We also show that the semidiscrete blow-up time converges to the theoretical one when the mesh size goes to zero. A similar study has been undertaken in [1] and [6] where the authors have considered the problem (1)–(3) in the case where the reaction term $(u(0,t))^p$ is replaced by $(u(x,t))^p$. In the same way in [2] the numerical extinction has been studied using some discrete and semidiscrete schemes (a solution u extincts in a finite time if it reaches the value zero in a finite time).

Our paper is written in the following manner. In the next section, we prove some results about the discrete maximum principle. In the third section, we show that the solution of the semidiscrete problem blows up in a finite time and estimate its semidiscrete blow-up time. In the fourth section, we give a result about the convergence of the semidiscrete blow-up time in some cases where the blow-up occurs. Finally, in the last section, we give some numerical results to illustrate our analysis.

2 Properties of the semidiscrete scheme

In this section, we give some lemmas which will be used later.
The following lemma is a semidiscrete version of the maximum principle.

Lemma 2.1 *Let $a_h(t) \in C^0([0,T], \mathbb{R}^{I+1})$ and let $V_h(t) \in C^1([0,T], \mathbb{R}^{I+1})$ such that*

$$\frac{d}{dt}V_i(t) - \delta^2 V_i(t) + a_k(t)V_k(t) \geq 0, \quad 1 \leq i \leq I-1, \quad t \in (0,T), \tag{7}$$

$$V_0(t) \geq 0, \quad V_I(t) \geq 0, \quad t \in (0,T), \tag{8}$$

$$V_i(0) \geq 0, \quad 0 \leq i \leq I. \tag{9}$$

Then we have $V_i(t) \geq 0$, $0 \leq i \leq I$, $t \in (0,T)$.

Proof. Let $T_0 < T$ and let $m = \min_{0\leq i\leq I, 0\leq t\leq T_0} V_i(t)$. Since for $i \in \{0, ..., I\}$, $V_i(t)$ is a continuous function, there exists $t_0 \in [0, T_0]$ such that $m = V_{i_0}(t_0)$ for a certain $i_0 \in \{0, ..., I\}$. Assume that $m < 0$. If $i_0 = 0$ or $i_0 = I$, we have a contradiction because of (8). For $i_0 \in \{1, ..., I-1\}$, it is not hard to see that

$$\frac{dV_{i_0}(t_0)}{dt} = \lim_{k\to 0} \frac{V_{i_0}(t_0) - V_{i_0}(t_0 - k)}{k} \leq 0, \tag{10}$$

$$\delta^2 V_{i_0}(t_0) = \frac{V_{i_0+1}(t_0) - 2V_{i_0}(t_0) + V_{i_0-1}(t_0)}{h^2} \geq 0. \tag{11}$$

Define the vector $Z_h(t) = e^{\lambda t} V_h(t)$ where λ is large enough that $a_k(t_0)V_k(t_0) - \lambda m < 0$. Use (10) and (11) to obtain $\frac{dZ_{i_0}(t_0)}{dt} \leq 0$ and $\delta^2 Z_{i_0}(t_0) \geq 0$, which implies that

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + e^{\lambda t_0}(a_k(t_0)V_k(t_0) - \lambda m) < 0. \tag{12}$$

On the other hand, from (7), we derive the following inequality

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + e^{\lambda t_0}(a_k(t_0)V_k(t_0) - \lambda m) \geq 0.$$

Therefore, we have a contradiction because of (12).
Another form of the maximum principle for semidiscrete equations is the comparison lemma below.

Lemma 2.2 *Let $V_h(t)$, $U_h(t) \in C^1([0, T], \mathbb{R}^{I+1})$ and $f \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ such that for $t \in (0, T)$*

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + f(V_k(t), t) \geq \frac{dU_i(t)}{dt} - \delta^2 U_i(t) + f(U_k(t), t),\ 1 \leq i \leq I-1, \tag{13}$$

$$V_0(t) \geq U_0(t), \quad V_I(t) \geq U_I(t), \tag{14}$$

$$V_i(0) \geq U_i(0), \quad 0 \leq i \leq I. \tag{15}$$

Then we have $V_i(t) \geq U_i(t)$, $0 \leq i \leq I$, $t \in (0, T)$.

Proof. Introduce the vector $Z_h(t) = V_h(t) - U_h(t)$. A direct calculation yields

$$\frac{dZ_i(t)}{dt} - \delta^2 Z_i(t) + f_y(\theta_k(t), t) Z_k(t) \geq 0,$$

$$Z_0(t) \geq 0, \quad Z_I(t) \geq 0,$$

$$Z_i(0) \geq 0,$$

where θ_k is an intermediate value between U_k and V_k and f_y is the partial derivative of f with respect to the second variable. Since $f \in C^1$ then $f_y(\theta_k(t), t)$ is bounded on $(0, T)$. Use Lemma 2.1 to complete the rest of the proof.
The lemma below shows that when i is between 1 and $I-1$, then $U_i(t)$ is positive where $U_h(t)$ is the solution of the semidiscrete problem.

Lemma 2.3 *Let U_h be the solution of (4)–(6). Then we have*

$$U_i(t) > 0, \quad 1 \leq i \leq I-1.$$

Proof. Let $\alpha = \min_{1\leq i\leq I-1} \varphi_i$ and introduce the vector V_h defined by $V_i = \alpha e^{-\lambda_h t}\sin(i\pi h)$, $0 \leq i \leq I$, where $\lambda_h = \frac{2-2\cos(h\pi)}{h^2}$. It is not hard to see that

$$\frac{dU_i}{dt} - \delta^2 U_i \geq \frac{dV_i}{dt} - \delta^2 V_i = 0,$$

$$U_0(t) = V_0(t), \quad U_I(t) = V_I(t) = 0,$$

$$U_i(0) \geq V_i(0), \quad 1 \leq i \leq I-1.$$

We deduce from Lemma 2.2 that $U_i(t) \geq \alpha e^{-\lambda_h t}\sin(i\pi h)$, $0 \leq i \leq I$. This implies that $U_i(t) > 0$, $1 \leq i \leq I-1$, and the proof is complete.
The following lemma reveals that the solution $U_h(t)$ of the semidiscrete problem is symmetric and $\delta^+ U_i(t)$ is positive when i is between 1 and $k-1$.

Lemma 2.4 *Let U_h be the solution of (4)–(6). Then we have for $t \in (0, T_b^h)$*

$$U_{I-i}(t) = U_i(t), \ 0 \leq i \leq I, \quad \delta^+ U_i(t) > 0, \ 0 \leq i \leq k-1. \tag{16}$$

Proof. Introduce the vector V_h defined as follows $V_i(t) = U_{I-i}(t)$ for $0 \leq i \leq I$. It is not hard to see that $V_h(t)$ is a solution of (4)–(6). It follow from Lemma 2.2 that $V_h(t) = U_h(t)$. Now, define the vector $Z_h(t)$ such that

$$Z_i(t) = U_{i+1}(t) - U_i(t), \quad 0 \leq i \leq k-1,$$

and let t_0 be the first $t > 0$ such that $Z_i(t) > 0$ for $t \in [0, t_0)$ but $Z_{i_0}(t_0) = 0$. Without loss of the generality, we assume that i_0 is the smallest integer which guarantees the equality. If $i_0 = 0$ then we have $U_1(t_0) = U_0(t_0) = 0$, which is a contradiction because from Lemma 2.3, $U_1(t_0) > 0$. It is easy to see that

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) = 0, \quad if \quad 1 \leq i_0 \leq k-1. \tag{17}$$

On the other hand, we observe that

$$\frac{dZ_{i_0}(t_0)}{dt}=\lim_{k\to 0}\frac{Z_{i_0}(t_0)-Z_{i_0}(t_0-k)}{k}\leq 0,$$

$$\delta^2 Z_{i_0}(t_0)=\frac{Z_{i_0+1}(t_0)-2Z_{i_0}(t_0)+Z_{i_0-1}(t_0)}{h^2}>0,\quad 1\leq i_0\leq k-2,$$

and we know that if $i_0=k-1$,

$$\begin{aligned}\delta^2 Z_{k-1}(t_0) &= \delta^2 U_k(t_0)-\delta^2 U_{k-1}(t_0)\\ &= \frac{U_{k+1}(t_0)-2U_k(t_0)+U_{k-1}(t_0)-U_k(t_0)+2U_{k-1}(t_0)-U_{k-2}(t_0)}{h^2}.\end{aligned}$$

Since k is the integer part of the number $I/2$, using the fact that the discrete solution is symmetric, we have either $U_{k+1}(t)=U_{k-1}(t)$ or $U_{k+1}(t)=U_k(t)$. In both cases, we find that

$$\delta^2 Z_{k-1}(t_0)=\frac{Z_{k-2}(t_0)}{h^2}>0.$$

The above inequalities imply that $\frac{dZ_{i_0}(t_0)}{dt}-\delta^2 Z_{i_0}(t_0)<0$, which is a contradiction because of (17) and the proof is complete.
A discrete version of the Green's formula is the following

Lemma 2.5 *Let $U_h(t)$ and $V_h(t)$ two vectors such that $U_0(t)=0$, $U_I(t)=0$, $V_0(t)=0$, $V_I(t)=0$. Then we have*

$$\sum_{i=1}^{I-1} hU_i\delta^2 V_i=\sum_{i=1}^{I-1} hV_i\delta^2 U_i. \tag{18}$$

Proof. A routine calculation yields

$$\sum_{i=1}^{I-1} hU_i\delta^2 V_i=\sum_{i=1}^{I-1} hV_i\delta^2 U_i+\frac{V_I U_{I-1}-U_I V_{I-1}+V_0U_1-U_0V_1}{h}$$

and the result follows using the assumptions of the lemma.
To end this section, let us state a result on the operator δ^2.

Lemma 2.6 *Let $U_h\in C^0([0,T],R^{I+1})$ such that $U_h\geq 0$. Then we have*

$$\delta^2 U_i^p\geq pU_i^{p-1}\delta^2 U_i\quad for\quad 1\leq i\leq I-1.$$

Proof. Using Taylor's expansion, we get

$$\delta^2 U_i^p=pU_i^{p-1}\delta^2 U_i+(U_{i+1}-U_i)^2\frac{p(p+1)}{2h^2}\theta_i^{p-2}+(U_{i-1}-U_i)^2\frac{p(p+1)}{2h^2}\eta_i^{p-2}\quad if\quad 1\leq i\leq I-1,$$

where θ_i is an intermediate value between U_i and U_{i+1} and η_i the one between U_i and U_{i-1}. The result follows taking into account the fact that U_h is nonnegative.

3 Blow-up solutions

In this section, under some assumptions, we show that the solution of the semidiscrete problem blows up in a finite time and estimate its semidiscrete blow-up time.
The statement of our first result on blow-up is the following

Theorem 3.1 *Suppose that there exists a positive constant A such that the initial data at (6) obeys*

$$\delta^2\varphi_i + \varphi_i \geq A\varphi_i^p, \quad 0 \leq i \leq I. \tag{19}$$

Then the solution U_h blows up in a finite time T_b^h with the following estimation

$$T_b^h \leq \frac{1}{A}\frac{\|\varphi_h\|_\infty^{1-p}}{(p-1)}. \tag{20}$$

Proof. Since $(0, T_b^h)$ is the maximal time interval on which $\|U_h(t)\|_\infty$ is finite, our aim is to show that T_b^h is finite and satisfies the above inequality. Introduce the vector $J_h(t)$ defined as follows

$$J_i = \frac{d}{dt}U_i - AU_i^p, \quad 0 \leq i \leq I. \tag{21}$$

A direct calculation yields

$$\frac{d}{dt}J_i - \delta^2 J_i = \frac{d}{dt}(\frac{d}{dt}U_i - \delta^2 U_i) - ApU_i^{p-1}\frac{d}{dt}U_i + A\delta^2 U_i^p.$$

From Lemma 2.6, $\delta^2 U_i^p \geq pU_i^{p-1}\delta^2 U_i$ which implies that

$$\frac{d}{dt}J_i - \delta^2 J_i \geq \frac{d}{dt}(\frac{d}{dt}U_i - \delta^2 U_i) - ApU_i^{p-1}(\frac{d}{dt}U_i - \delta^2 U_i),\ 1 \leq i \leq I-1.$$

Use (4) to obtain

$$\begin{aligned}\frac{d}{dt}J_i - \delta^2 J_i &\geq pU_k^{p-1}\frac{d}{dt}U_k - ApU_i^{p-1}U_k^p \\ &\geq pU_k^{p-1}(J_k + AU_k^p) - ApU_i^{p-1}U_k^p \\ &\geq pU_k^{p-1}J_k + AU_k^p(U_k^{p-1} - U_i^{p-1}).\end{aligned}$$

From Lemma 2.4, $U_k \geq U_i$. We deduce that

$$\frac{d}{dt}J_i - \delta^2 J_i \geq pU_k^{p-1}J_k, \quad 1 \leq i \leq I-1.$$

Obviously, we have $J_0(t) = 0$, $J_I(t) = 0$ and the relation (19), implies that $J_h(0) \geq 0$. It follows from Lemma 2.1 that $J_h(t)$ is nonnegative, which implies $\frac{d}{dt}U_i \geq AU_i^p$, $0 \leq i \leq I$. We observe that

$$\frac{dU_i}{U_i^p} \geq Adt, \quad 0 \leq i \leq I. \tag{22}$$

Integrating this inequality over (t, T_b^h), we arrive at

$$T_b^h - t \leq \frac{1}{A}\frac{(U_i(t))^{1-p}}{(p-1)}, \tag{23}$$

Since $\|U_h(t)\|_\infty = U_k(t)$, if we replace i by k and the time t by 0, we get the following estimation $T_b^h \leq \frac{1}{A}\frac{\|U_h(0)\|_\infty^{1-p}}{(p-1)}$, which implies that T_b^h is finite and verifies (20). Thus the proof is complete.

Remark 3.1 *The inequality (23) implies that*

$$T_b^h - t_0 \leq \frac{1}{A}\frac{\|U_h(t_0)\|_\infty^{1-p}}{(p-1)} \quad \text{if} \quad 0 < t_0 < T_b^h.$$

The proof of the above theorem allows us to establish the estimation in Remark 3.1 which is crucial to prove the convergence of the semidiscrete blow-up time. Unfortunately because of the restriction on the initial data, this theorem is not optimal to determine blow-up solutions. The following theorem is more acceptable to have blow-up solutions.

Theorem 3.2 *Assume that the initial data at (4) satisfies $v(0) > \lambda_h^{\frac{1}{p-1}}$ where*

$$v(0) = \sum_{i=1}^{I-1} \tan(\frac{\pi}{2}h)\sin(i\pi h)\varphi_i,$$

with $\lambda_h = \frac{2-2\cos(\pi h)}{h^2}$.
Then the solution $U_h(t)$ of (4)–(6) blows up in a finite time T_b^h which is estimated as follows

$$T_b^h \leq \frac{1}{(p-1)((v(0))^{p-1} - \lambda_h)}.$$

Proof. Since $(0, T_b^h)$ is the maximal time interval on which $\|U_h(t)\|_\infty$ is finite, our goal is to prove that T_b^h is finite and obeys the above inequality. Introduce the function $v(t)$ defined as follows

$$v(t) = \sum_{i=1}^{I-1} \tan(\frac{\pi}{2}h)\sin(i\pi h)U_i(t).$$

Take the derivative of v with respect to t and use (4) to obtain

$$v^{'}(t) = \sum_{i=1}^{I-1} \tan(\frac{\pi}{2}h)\sin(i\pi h)(\delta^2 U_i(t) + U_k^p(t)).$$

We observe that $\delta^2 \sin(i\pi h) = -\lambda_h \sin(i\pi h)$. From the above equality and Lemma 2.5, we arrive at

$$v^{'}(t) = -\lambda_h v(t) + U_k^p(t)\sum_{i=1}^{I-1} \tan(\frac{\pi}{2}h)\sin(i\pi h).$$

By a routine calculation, we find that $\sum_{i=1}^{I-1}\tan(\frac{\pi}{2}h)\sin(i\pi h)$ equals one. Due to the fact that U_k^p is bigger than $v^p(t)$, we get $v^{'}(t) \geq v^p(t)(1 - \frac{\lambda_h}{v^{p-1}(t)})$. It is easy to see that $v(t)$ is an increasing function. Indeed, let t_0 be the first $t > 0$ such that $v^{'}(t) > 0$ for $t \in [0, t_0)$ but $v^{'}(t_0)$. This implies that $0 \geq v^p(t_0)(1 - \frac{\lambda_h}{v^{p-1}(t_0)}) > 0$, which is impossible. Hence $v^{'}(t) > 0$ for $t \in [0, T_b^h)$. Therefore we have $v(t) \geq v(0)$ and $v^{'}(t) \geq v^p(t)(1 - \frac{\lambda_h}{v^{p-1}(0)})$. The above inequality may be rewritten in the following form $\frac{dv}{v^p} \geq (1 - \frac{\lambda_h}{v^{p-1}(0)})dt$. Integrating this inequality over $(0, T_b^h)$, we find $T_b^h \leq \frac{1}{(p-1)((v(0))^{p-1} - \lambda_h)}$. We conclude that T_b^h is finite and the proof is complete.

Remark 3.2 *Since $U_k(t) = \|U_h(t)\|_\infty$, it is easy to see that*

$$\delta^2 U_k(t) = \frac{U_{k+1}(t) - 2U_k(t) + U_{k-1}(t)}{h^2} \leq 0.$$

Therefore, using (4), we get $\frac{dU_k}{dt} \geq U_k^p$ which implies that $\frac{dU_k}{U_k^p} \geq dt$. Integrating this inequality over $(0, T_b^h)$, we arrive at $T_b^h \geq \frac{\|U_h(0)\|_\infty^{1-p}}{p-1}$. Thus we have a lower bound of the semidiscrete blow-up time.

Remark 3.3 *Consider the following semidiscrete scheme*

$$\frac{d}{dt}V_i(t) = \delta^2 V_i(t) + V_i^p(t), \quad 1 \leq i \leq I-1, \quad t \in (0, T_h),$$

$$V_0(t) = 0, \quad V_I(t) = 0, \quad t \in (0, T_h),$$

$$V_i(0) = \varphi_i, \quad 0 \leq i \leq I,$$

where $(0, T_h)$ is the maximal time interval on which $\|V_h(t)\|_\infty$ is finite. We observe that the above scheme is a semidiscretization of the continuous problem below

$$u_t(x,t) = u_{xx}(x,t) + (u(x,t))^p, \quad (x,t) \in (-l, l) \times (0, T),$$

$$u(-l,t)=0,\quad u(l,t)=0,\quad t\in(0,T),$$

$$u(x,0)=u_0(x)\geq 0,\quad x\in(-l,l).$$

Let $U_h(t)$ be the solution of (4)–(6). We know from Lemma 2.4 that $U_k\geq U_i$ for $1\leq i\leq I-1$, which implies that

$$\frac{d}{dt}U_i(t)\geq\delta^2U_i(t)+U_i^p(t),\quad 1\leq i\leq I-1,\quad t\in(0,T_b^h),$$

$$U_0(t)=0,\quad U_I(t)=0,\quad t\in(0,T_b^h),$$

$$U_i(0)=\varphi_i,\quad 0\leq i\leq I.$$

Setting $Z_h(t)=U_h(t)-V_h(t)$, it is not hard to see that

$$\frac{d}{dt}Z_i-\delta^2Z_i-p\xi_i^{p-1}Z_i\geq 0,\quad 1\leq i\leq I-1,\quad t\in(0,T_h^*),$$

$$Z_0(t)=0,\quad Z_I(t)=0,\quad t\in(0,T_h^*),$$

$$Z_i(0)=0,\quad 0\leq i\leq I,$$

where $T_h^*=\min\{T_h,T_b^h\}$, ξ_i is an intermediate value between $U_i(t)$ and $V_i(t)$. Modifying slightly the proof of Lemma 2.1, we find that $Z_h(t)\geq 0$ for $t\in(0,T_h^*)$. In other words, we have $U_h(t)\geq V_h(t)$ for $t\in(0,T_h^*)$ and we conclude that $T_b^h\leq T_h$.

4 Convergence of semidiscrete blow-up times

In this section, under some assumptions, we show that the semidiscrete blow-up time for the solution of the semidiscrete problem converges to the real one when the mesh size goes to zero. In order to prove this result, firstly, we prove the convergence of the semidiscrete scheme by the following

Theorem 4.1 *Assume that (1)-(3) has a solution $u\in C^{4,1}([0,1]\times[0,T])$ and the initial condition at (6) satisfies*

$$\|\varphi_h-u_h(0)\|_\infty=o(1)\quad as\quad h\to 0,\tag{24}$$

where $u_h(t)=(u(x_0,t),...,u(x_I,t))^T$. Then, for h sufficiently small, the problem (4)–(6) has a unique solution $U_h\in C^1([0,T],\mathbb{R}^{I+1})$ such that

$$\max_{0\leq t\leq T}\|U_h(t)-u_h(t)\|_\infty=O(\|\varphi_h-u_h(0)\|_\infty+h^2)\quad as\quad h\to 0.\tag{25}$$

Proof. Since $u \in C^{4,1}$, there exist two positive constants K and M such that

$$\frac{\|u_{xxxx}\|_\infty}{12} \leq K, \quad \|u\|_\infty \leq K, \quad p(K+1)^{p-1} \leq M. \tag{26}$$

The problem (4)–(6) has for each h, a unique solution $U_h \in C^1([0, T_b^h), \mathbb{R}^{I+1})$. Let $t(h)$ the greatest value of $t > 0$ such that

$$\|U_h(t) - u_h(t)\|_\infty < 1 \quad for \quad t \in (0, t(h)). \tag{27}$$

The relation (24) implies that $t(h) > 0$ for h sufficiently small. Let $t^*(h) = \min\{t(h), T\}$. Use the triangle inequality to obtain

$$\|U_h(t)\|_\infty \leq \|u(x,t)\|_\infty + \|U_h(t) - u_h(t)\|_\infty \quad for \quad t \in (0, t^*(h)).$$

From (26) and (27), we see that $\|U_h(t)\|_\infty$ is bounded from above by the constant $1 + K$. Let $e_h(t) = U_h(t) - u_h(x,t)$ be the error of discretization. Using Taylor's expansion, we find that

$$\frac{d}{dt}e_i(t) - \delta^2 e_i(t) = \frac{h^2}{12}u_{xxxx}(\widetilde{x}_i, t) + p\xi_k^{p-1}e_k(t), \quad 1 \leq i \leq I-1, \quad t \in (0, t^*(h)),$$

where ξ_k is an intermediate value between $U_k(t)$ and $u(x_k, t)$. From (26), we derive the following inequality

$$\frac{d}{dt}e_i(t) - \delta^2 e_i(t) \leq p\xi_k^{p-1}e_k(t) + Kh^2, \quad 1 \leq i \leq I-1, \quad t \in (0, t^*(h)). \tag{28}$$

Let z_h the vector defined by

$$z_i = e^{(M+1)t}(\|\varphi_h - u_h(0)\|_\infty + Kh^2), \quad 0 \leq i \leq I.$$

Since $U_k(t)$ is bounded from above by $1 + K$ and ξ_k is an intermediate value between $U_k(t)$ and $u(x_k, t)$, from (26), we see that M is bigger than $p\xi_k^{p-1}$. Hence by a straightforward computation, we observe that

$$\frac{d}{dt}z_i - \delta^2 z_i > p\xi_k^{p-1}z_k(t) + Kh^2, \quad 1 \leq i \leq I-1, \quad t \in (0, t^*(h)),$$

$$z_0 > e_0, \quad z_I > e_I,$$

$$z_i(0) > e_i(0), \quad 0 \leq i \leq I.$$

It follows from Lemma 2.2 that $z_h \geq e_h(t)$ for $t \in (0, t^*(h))$. By the same way, we also prove that $z_h \geq -e_h(t)$ for $t \in (0, t^*(h))$, which implies that

$$\|U_h(t) - u_h(t)\|_\infty \leq e^{(M+1)t}(\|\varphi_h - u_h(0)\|_\infty + Kh^2), \quad t \in (0, t^*(h)).$$

Let us show that $t^*(h) = T$. Suppose that $T > t(h)$. From (27), we obtain

$$1 = \|U_h(t(h)) - u_h(t(h))\|_\infty \leq e^{(M+1)T}(\|\varphi_h - u_h(0)\|_\infty + Kh^2). \qquad (29)$$

Since the term on the right hand side of the inequality in (29) goes to zero as h tends to zero, we deduce that $1 \leq 0$, which is impossible. Consequently $t^*(h) = T$, and we obtain the desired result.
Now, we are in a position to prove the main theorem of this section.

Theorem 4.2 *Suppose that the problem (1)–(3) has a solution u which blows up in a finite time T_b such that $u \in C^{4,1}([0,1]\times[0,T_b))$ and the initial condition at (6) satisfies*

$$\|\varphi_h - u_h(0)\|_\infty = o(1) \quad as \quad h \to 0.$$

Under the assumption of Theorem 3.1, the problem (4)–(6) has a solution $U_h(t)$ which blows up in a finite time T_b^h and $\lim_{h\to 0} T_b^h = T_b$.

Proof. Let $\varepsilon > 0$. There exists a positive constant N such that

$$\frac{1}{A}\frac{x^{1-p}}{(p-1)} \leq \frac{\varepsilon}{2} \quad for \quad x \in [N, +\infty). \qquad (30)$$

Since u blows up at the time T_b, there exists a time T_1 such that $|T_1 - T_b| \leq \frac{\varepsilon}{2}$ and $\|u(x,t)\|_\infty \geq 2N$ for $t \in [T_1, T_b)$. Letting $T_2 = \frac{T_1+T_b}{2}$, we see that u is bounded on the interval $[0, T_2]$. It follows from Theorem 4.1 that the problem (4)–(6) has a solution $U_h(t)$ which obeys $\sup_{t\in[0,T_2]} \|U_h(t) - u_h(t)\|_\infty \leq N$. Applying the triangle inequality, we get $\|U_h(t)\|_\infty \geq \|u_h(t)\|_\infty - \|U_h(t) - u_h(t)\|_\infty$, which leads to $\|U_h(t)\|_\infty \geq N$ for $t \in [0, T_2]$. From Theorem 3.1, $U_h(t)$ blows up at the time T_b^h. We deduce from Remark 3.1 and (30) that

$$|T_b - T_b^h| \leq |T_b - T_2| + |T_b^h - T_2| \leq \frac{\varepsilon}{2} + \frac{1}{A}\frac{\|U_h(T_2)\|_\infty^{1-p}}{(p-1)} \leq \varepsilon,$$

and we have the desired result.

5 Numerical results

In this section, we consider the following explicit scheme.

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n^e} = \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} + (U_k^{(n)})^p, \quad 1 \leq i \leq I-1,$$

$$U_0^{(n)} = 0, \quad U_I^{(n)} = 0,$$

$$U_i^0 = 20 * \sin(\pi h i), \quad 0 \le i \le I,$$

and the following implicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{h^2} + (U_k^{(n)})^p, \quad 1 \le i \le I-1,$$

$$U_0^{(n+1)} = 0, \quad U_I^{(n+1)} = 0,$$

$$U_i^0 = 20 * \sin(\pi h i), \quad 0 \le i \le I,$$

where $n \ge 0$, $k = \frac{I}{2}$, $\Delta t_n = \frac{h^2}{\|U_h^{(n)}\|_\infty^{p-1}}$, $\Delta t_n^e = \min\{\frac{h^2}{2}, \Delta t_n\}$, $T^n = \sum_{j=0}^{n-1} \Delta t_j$. In the following tables, in rows, we present the numerical blow-up times, the numbers of iterations, CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128, 256. The numerical blow-up time $T^n = \sum_{j=0}^{n-1} \Delta t_j$ is computed at the first time when $|T^{n+1} - T^n| \le 10^{-16}$. The order(s) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

First case: $p = 2$.

Table 1: Numerical blow-up times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPU time	s
16	0.061364	7343	-	-
32	0.061150	27905	-	-
64	0.061097	105895	1	2.02
128	0.061083	400822	6	1.93
256	0.061080	1512394	44	2.23

Table 2: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	CPU time	s
16	0.061385	7343	-	-
32	0.061155	27905	-	-
64	0.061098	105895	3	2.02
128	0.061084	400822	19	2.03
256	0.061080	1512394	151	1.81

Second case: $p = 3$.

Table 3: Numerical blow-up times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPU time	s
16	0.012702	3248	-	-
32	0.012645	12261	-	-
64	0.012632	46184	-	2.14
128	0.012629	173362	3	2.12
256	0.012628	648006	22	1.59

Table 4: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	CPU time	s
16	0.012701	3248	-	-
32	0.012643	12261	-	-
64	0.012630	46184	1	2.16
128	0.012627	173362	9	2.12
256	0.012626	648006	65	1.59

From Remark 3.3, we have seen that the blow-up time of the solution of our problem is smaller that the one of the problem where the reaction term is not local. In order to verify this assertion, we do the same experiments when the reaction term is not local and is replaced by $(U_i^{(n)})^p$

First case: $p = 2$.

Table 5: Numerical blow-up times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPU time	s
16	0.082331	7658	-	-
32	0.082411	29260	-	-
64	0.082431	111589	1	2.01
128	0.082436	424450	6	2.01
256	0.082437	1609727	42	2.33

Table 6: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	CPU time	s
16	0.082447	7659	-	-
32	0.082440	29261	-	-
64	0.082438	111590	3	1.81
128	0.082437	424451	21	1.00
256	0.082436	1609728	157	1.00

Second case: $p = 3$.

Table 7: Numerical blow-up times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPU time	s
16	0.012861	3257	-	-
32	0.012827	12314	-	-
64	0.012819	46455	-	2.09
128	0.012817	174655	4	2.01
256	0.012809	653960	19	2.01

Table 8: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	CPU time	s
16	0.012863	3257	-	-
32	0.012829	12314	1	-
64	0.012821	46455	2	2.09
128	0.012819	174655	9	2.01
256	0.012818	653960	64	1.00

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