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Reproducing Kernel Method for Solving Systems of Linear Equations

Yonggang Ye

School of Basic Science, Harbin University of Commerce
Harbin, Heilongjiang 150028, China

Fazhan Geng[1]

Department of Mathematics, Changshu Institute of Technology
Changshu, Jiangsu 215500, China

Abstract

A new method for finding the exact solutions of systems of linear equations is presented. Advantage of this method is the simplicity of the procedure. There are no additional constraint conditions. The method can avoid evaluation of determinants and matrix computation, and this reduces the amount of computation. Also, the method is valid when the coefficient matrix of the linear system is singular and the solution of the the system exists, and the solution obtained using our method is the minimal norm least-squares solution. Some numerical examples are studied to test the presented method.

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1 Introduction

Here, we consider the linear system

$$Ax = b, A = (a_{ij}) \in R^{n\times n}, x = (x_i) \in R^n, b = (b_i) \in R^n \tag{1}$$

where A is a nonsingular matrix.

Systems of linear equations are very important in application of mathematics. These systems arise in many different scientific applications. Notably,

[1]Corresponding author
E-mail address: gengfazhan@sina.com(Fazhan geng)

partial differential equations discretized with finite difference or finite element methods yield systems of linear equations. As we know, Gramer's Rule, Gaussian elimination[1] and LU-factorization[2-5] can be used to find the exact solution of a system of linear equations. Also, there are other methods of obtaining approximate solutions of linear system. However, Gramer's Rule needs abundant evaluation of determinants.

In this paper, we will give a new method for finding the exact solution of system(1.1) using the reproducing property of reproducing kernel. The presented method can reduce the amount of computation and need not additional constraint conditions. Moreover, the method is valid when the coefficient matrix A is singular and the solution of system(1.1) exists, and the solution obtained using our method is the minimal norm least-squares solution, that is, $\sum\limits_{k=1}^{n} x_k^2$ is minimal.

Regard x, b as the constant functions of t defined on $[a, b]$(a, b are arbitrary), then system(1.1) can be converted into the following system of function equations

$$Ax(t) = b(t) \tag{2}$$

Clearly, $x,\ b \in \bigoplus\limits_{n} W_2^1[a,b]$, $A = (A_{ij})$ and $A_{ij}x_j(t) = a_{ij}x_j(t)$, $i, j = 1, 2, \cdots, n$. $W_2^1[a,b]$ and $\bigoplus\limits_{n} W_2^1[a,b]$ are defined in the following section.

2 Some preliminaries

2.1 The reproducing kernel space $W_2^1[a,b]$

The inner product space $W_2^1[a,b]$ is defined by $W_2^1[a,b] = \{u(x) \mid u$ is absolutely continuous real valued function, $u, u' \in L^2[a,b]\}$. The inner product and norm in $W_2^1[a,b]$ are given respectively by

$$(u(x), v(x))_{W_2^1} = \int_a^b (uv + u'v')dx, \ \| u \|_{W_2^1} = \sqrt{(u,u)_{W_2^1}},$$

where $u(x), v(x) \in W_2^1[a,b]$. In [6], the authors proved that $W_2^1[a,b]$ is a reproducing kernel space and its reproducing kernel is

$$R_x(y) = \frac{1}{2\sinh(b-a)}[\cosh(x+y-b-a) + \cosh(|x-y|-b+a)].$$

2.2 The reproducing kernel space $\bigoplus\limits_{n} W_2^1[a,b]$

The space $\bigoplus\limits_{n} W_2^1[a,b]$ is defined as $\bigoplus\limits_{n} W_2^1[a,b] = \{u = (u_1, u_2, \cdots, u_n)^\top | u_i \in W_2^1[a,b], i = 1, 2, \cdots, n\}$. The inner product and norm are given respectively by $(u,v)=\sum\limits_{i=1}^{n}(u_i,v_i)_{W_2^1}$, $\| u \|=(\sum\limits_{i=1}^{n}\| u_i \|^2)^{\frac{1}{2}}, u,v\in\bigoplus\limits_n W_2^1[a,b]$. Clearly, $\bigoplus\limits_n W_2^1[a,b]$ is a Hilbert space.

2.3 Important Lemma

Lemma 2.1. *If $A_{ij}:W_2^1[a,b]\to W_2^1[a,b], i,j=1,2,\cdots,n$ are bounded linear operators, then $A:\bigoplus\limits_n W_2^1[a,b]\to\bigoplus\limits_n W_2^1[a,b]$ is also a bounded linear operator.*

Proof. Clearly, A is a linear operator. For $\forall\ u\in\bigoplus\limits_n W_2^1[a,b]$,

$$\begin{aligned}\| Au \| &=(\sum_{i=1}^{n}\|\sum_{j=1}^{n}A_{ij}u_j\|^2)^{\frac{1}{2}}\\ &\le[\sum_{i=1}^{n}(\sum_{j=1}^{n}\| A_{ij}\|\ \| u_j\|)^2]^{\frac{1}{2}}\\ &\le[\sum_{i=1}^{n}(\sum_{j=1}^{n}\| A_{ij}\|^2)(\sum_{j=1}^{n}\| u_j\|^2)]^{\frac{1}{2}}\\ &=(\sum_{i=1}^{n}\sum_{j=1}^{n}\| A_{ij}\|^2)^{\frac{1}{2}}\| u\|.\end{aligned} \tag{1}$$

The boundedness of A_{ij} implies that A is bounded. The proof is complete. □

It is easy to see that the adjoint operator of A is $A^*=\begin{pmatrix}A_{11}^* & A_{21}^* & \cdots & A_{n1}^*\\ A_{12}^* & A_{22}^* & \cdots & A_{n2}^*\\ \vdots & \vdots & \ddots & \vdots\\ A_{1n}^* & A_{2n}^* & \cdots & A_{nn}^*\end{pmatrix}$, where A_{ij}^* is the adjoint operator of A_{ij}.

3 The method for solving system(1.1)

In this section, we will give the representation of exact solution of system(1.1) and the concrete implementation method.

In view of Lemma(2.1), it is clear that $A:\bigoplus\limits_n W_2^1[a,b]\to\bigoplus\limits_n W_2^1[a,b]$ is a bounded linear operator. Take a point t_0 on $[a,b]$ arbitrarily. Put $\varphi_j(x)=R_{t_0}(x)\overrightarrow{e_j}=(0,0,...,\underbrace{R_{t_0}(x)}_{j},0,...,0)^\top$ and $\psi_j(x)=A^*\varphi_j(x)$, $j=1,2,\cdots,n$, where $R_x(y)$ is the reproducing kernel of $W_2^1[a,b]$ and A^* is the adjoint operator of A . The orthonormal system $\{\overline{\psi}_j(x)\}_{j=1}^n\in\bigoplus\limits_n W_2^1[a,b]$ can be derived from Gram-Schmidt orthogonalization process of $\{\psi_j(x)\}_{j=1}^n$,

$$\overline{\psi}_j(x)=\sum_{k=1}^{j}\beta_{jk}\psi_k(x), j=1,2,\cdots,n. \tag{1}$$

Let

$$x(t) = \sum_{k=1}^{n} B_k \overline{\psi}_k(t) \tag{2}$$

, where $B_k = \sum_{l=1}^{k} \beta_{kl} b_l$. The following Theorem give the solution of system(1.1).

Theorem 3.1. *$x(t_0)$ is the solution of system(1.1).*

Proof. Note here that

$$Ax(t) = \sum_{k=1}^{n} B_k A\overline{\psi}_k(t) \tag{3}$$

and

$$\begin{aligned} (Ax)_i(t_0) &= \sum_{k=1}^{n} B_k (A\overline{\psi}_k(t), \varphi_i(t)) \\ &= \sum_{k=1}^{n} B_k (\overline{\psi}_k(t), A^*\varphi_i(t)) \\ &= \sum_{k=1}^{n} B_k (\overline{\psi}_k(t), \psi_i(t)). \end{aligned} \tag{4}$$

Here $(Ax)_i$ denotes the ith component. Therefore,

$$\begin{aligned} \sum_{l}^{i} \beta_{il}(Ax)_l(t_0) &= \sum_{k=1}^{n} B_k (\overline{\psi}_k(t), \sum_{l}^{i} \beta_{il}\psi_l(t)) \\ &= \sum_{k=1}^{n} B_k (\overline{\psi}_k(t), \overline{\psi}_i(t)) \\ &= B_i = \sum_{l=1}^{i} \beta_{il} b_l. \end{aligned} \tag{5}$$

Let $i = 1$, then $(Ax)_1(t_0) = b_1$.
Let $i = 2$, then $(Ax)_2(t_0) = b_2$.
In the same way, $(Ax)_k(t_0) = b_k, k = 3, 4, \cdots, n$.
Thus,

$$(Ax)_k(t_0) = b_k, k = 1, 2, \cdots, n.$$

That is,

$$Ax(t_0) = b. \tag{6}$$

This means that $x(t_0)$ satisfies system(1.1), i.e. $x(t_0)$ is the solution of system(1.1) and the proof of the theorem is complete. □

4 Numerical example

In this section, some numerical examples are studied to demonstrate the validity of the present method. Results obtained by the method are compared with the exact solution of each example and are found to be in good agreement with each other.

Example 1

Consider the following system of linear equations

$$Ax = b,$$

where A is a 50th order Hilbert matrix. The exact solution $x = (x_1, x_2, \cdots, x_{50})^\top$, the element $x_i = 1/i, i = 1, 2, \cdots, 50$. We obtain the solution of the system using our method. When we take m=4000 in mathematica program in Appendix A, the absolute error between the exact solution and the solution obtained using our method is nearly $0. \times 10^{-814}$(The error arise from the Roundoff error of computing machine).

Example 2

Consider the following singular system of linear equations

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ 0x_1 + x_2 + x_3 = 2 \\ x_1 + x_2 + x_3 = 1 \end{cases}$$

The exact solution $x = (x_1, x_2, x_3)^\top = (-1, 2-c, c)^\top$(c is a arbitrary constant), and the minimal norm least-squares solution of this system is $x = (-1, 1, 1)^\top$. Using our method, We can obtain the least-squares solution of the system. When we take m=100 in mathematica program in Appendix A, the absolute error between the exact minimal norm least-squares solution and the solution obtained using our method is nearly $0. \times 10^{-99}$(The error arise from the Roundoff error of computing machine).

Example 3

Consider the following singular system of linear equations

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 1 \\ x_1 + 2x_2 + 3x_3 + 4x_4 = 2 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 = 0 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 = 0 \end{cases}$$

The exact solution $x = (x_1, x_2, x_3, x_4)^\top = (2c_1 + c_2, 1 - 3c_1 - 2c_2, c_2, c_1)^\top(c_1, c_2$ are arbitrary constants), and the least-squares solution of this system is $x = (0.4, 0.3, 0.2, 0.1)^\top$. Using our method, We can obtain the minimal norm least-squares solution of the system. When we take m=100 in mathematica program in Appendix A, the absolute error between the exact minimal norm least-squares solution and the solution obtained using our method is nearly $0. \times 10^{-99}$(The error arise from the Roundoff error of computing machine).

5 Conclusion

A new simple method for solving systems of linear equations is developed, and the corresponding mathematica program of the algorithm is also given. Moreover, the method is valid when when the coefficient matrix A of the linear system is singular and the solution of the the system exists, and the solution obtained using our method is the minimal norm least-squares solution. From the above test examples, we can see that the method is valid.

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