The Two-Photon Jaynes-Cummings Model: A Coherent State Approach

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Abstract

It is shown, by conventional mean-field technics, that \( su(2) \otimes su(1,1) \) coherent states are a good choice for describing the two-photon Jaynes-Cummings model in the sense that results based on mean-field methods for those states are in good agreement with exact results. We also find that this agreement is much improved if the constant of motion of the model is implemented exactly instead of being implemented only in the average. This is done by projecting the \( su(2) \otimes su(1,1) \) coherent states onto the physical subspaces. It is seen that the Hamiltonian of the model is unbounded from below, i.e., it lacks a ground-state. We also present and study a stabilized version of the model such that the relevant physical features are preserved. Along the article, special attention is given to the description of the super-radiant phase transition exhibited by the model.

Keywords: Coherent states, Lie algebras, two-photon Jaynes-Cummings model

1 Introduction

The one-photon Jaynes-Cummings (JC) model [1, 2] is the prototype model of matter-radiation interaction. It describes an assembly of \( A \) two-level atoms interacting with a monochromatic radiation field, its Hamiltonian being given by

\[
H = \omega_f a^\dagger a + \omega_o S_z + \lambda \left( S_+ a + S_- a^\dagger \right),
\]

where \( S_z, S_\pm \) are the collective angular momentum operators of the assembly of atoms, satisfying the commutation relations of the \( su(2) \) Lie algebra, \( [S_z, S_\pm] = \)
$\pm S_\pm$ and $[S_+, S_-] = 2S_z$, and $a^\dagger, a$ are, respectively, creation and destruction operators of the photon, satisfying the boson commutation relation $[a, a^\dagger] = I$, $\omega_f$ is the energy of each photon, $\omega_a$ is the energy splitting between the relevant atomic states and $\lambda$ is a real coupling constant. This is an exactly solvable model [3, 4, 5]. Despite its simple form, it incorporates important quantum features such as the collapse and revival of atomic inversion [6] and the squeezing of the radiation field [7]. Moreover, this model can be realized experimentally in the $^{85}$Rb atom micromaser [8], in the $^{138}$Ba atom microlaser [9] and with spin polarized neutrons subject to a magnetic field [10]. Since its appearance, several different versions and generalizations of the JC model have been studied: multiphoton and density-dependent interactions [5, 11, 12]; $n$-level atoms and multimode radiation fields [13]; inclusion of Kerr-type nonlinearity, Stark-shift [4, 14] and spin-orbit interaction terms [5]; $q$-deformation of the ordinary creation and annihilation operators of the photon [15]. Atomic spatial motion and the influence of the classical gravitational field have also been considered [16]. In the present article we focus on the two-photon Jaynes-Cummings (2JC) model, the two-photon interaction version of (1):

$$H = \omega_f a^\dagger a + \omega_a S_z + \lambda \left( S_+ a^2 + S_- a^\dagger 2 \right).$$

This is also an exactly solvable model [5]. Many of the investigations made with the simple JC model have also been made for the 2JC model. Its dynamical evolution [17], with special attention to the collapse-and-revival phenomenon in the atomic inversion [18], is an example of such investigations. Also, since the squeezing of the electromagnetic field is a multiphoton process, the squeezing properties of the 2JC model have been intensively studied [19].

In the present article, we are only interested in the resonant case of (2), i.e., $\omega_a/2 = \omega_f = \omega$.

Several investigations show that coherent states [20] are good choices for trial states in variational approaches to the description of the behavior of a great variety of quantum systems. For example, they have been recently used to describe the ground-state and RPA energies of the two-level pairing model and of the Lipkin model [21, 22, 23]. Moreover, they also have been used by the present authors to investigate some ground-state properties of the Buck-Sukumar model [24]. In the same spirit, we now use coherent states to investigate some properties of the 2JC model.
2 Constants of motion and physical subspaces of the model

Since we are only interested in the resonant 2JC model, from now on we work with the Hamiltonian

\[ H = a^\dagger a + 2S_z + \lambda \left( S_+a^2 + S_-a^\dagger^2 \right). \]  

(3)

For simplicity, we make \( \omega = 1 \). This means that in the sequel all energies will be given in units of \( \omega \).

The operators \( S_z, S_\pm \) satisfy the commutation relations of the \( su(2) \) Lie algebra,

\[ [S_+, S_-] = 2S_z, \quad [S_z, S_\pm] = \pm S_\pm. \]  

(4)

The Casimir operator \( S^2 = S_+S_+ + S_\mp S_\mp \) is then a constant of motion and its eigenvalues \( S(S + 1) \) characterize the assembly of atoms described by \( H \) in the sense that \( 2S \) is the total number of atoms.

On the other hand, the operators

\[ K_0 = \frac{1}{2} \left( a^\dagger a + \frac{1}{2} \right), \quad K_+ = \frac{1}{2}a^\dagger^2, \quad K_- = \frac{1}{2}a^2 \]  

(5)

satisfy the commutation relations

\[ [K_+, K_-] = -2K_0, \quad [K_0, K_\pm] = \pm K_\mp, \]  

(6)

which are the well known commutation relations of the \( su(1, 1) \) Lie algebra. The \( su(1, 1) \) Casimir operator is given by \( Q = -K_+K_+ + K_0^2 \pm K_0 \) and its eigenvalues are of the form \( k(k - 1), \ k \in \mathbb{C} \), where \( k \) characterizes the irreducible representations. For the present boson realization, we find \( Q = -\frac{3}{16} \), and so this representation is the direct sum of two irreducible representations: one characterized by \( k = \frac{1}{4} \) and the other by \( k = \frac{3}{4} \). It is easy to conclude that the irreducible vector space associated with \( k = \frac{1}{4} \) \( (k = \frac{3}{4}) \) is the subspace spanned by the eigenstates of the number operator \( a^\dagger a \) with even (odd) eigenvalues. This was to be expected from the form of the operators (5).

The Hilbert space of the model is \( \mathcal{H} = \mathcal{H}_F \otimes \mathcal{H}_B \), where \( \mathcal{H}_F \) and \( \mathcal{H}_B \) are respectively the fermion (assembly of two-level atoms) and boson (photon system) Hilbert subspaces. An orthonormal basis of \( \mathcal{H} \) is formed by the kets

\[ |S, m\rangle \otimes |n\rangle, \]  

(7)

where \( |S, m\rangle \) are (normalized) eigenstates of \( S^2 \) and \( S_z \), i.e., \( S^2|S, m\rangle = S(S + 1)|S, m\rangle, \ S_z|S, m\rangle = m|S, m\rangle \), and \( |n\rangle \) are (normalized) eigenstates of the
boson number operator, \( a^\dagger a|n\rangle = n|n\rangle \), with \( S = 0, 1/2, 1, 3/2, \ldots \), \( m = -S, -S + 1, \ldots \), \( S \) and \( n = 0, 1, 2, \ldots \). From the above discussion, \( \mathcal{H} \) is the direct sum of two invariant subspaces:\( \mathcal{H}_e \), spanned by the kets
\[
|S, m\rangle \otimes |2n\rangle , \tag{8}
\]
and \( \mathcal{H}_o \), spanned by the kets
\[
|S, m\rangle \otimes |2n + 1\rangle \tag{9}.
\]
The operator
\[
C = a^\dagger a + 2S_z \tag{10}
\]
is a constant of motion, since \([C, (S^+ a^2 + S^- a^\dagger 2)] = 0\). The eigenvalues of \( C \), which we denote by \( c \), have the form \( c = -2S + r \), \( r \in \mathbb{N}_0 \), so that they run over the integers. The eigenvectors of \( C \) associated with the eigenvalue \( c \) are the kets
\[
|S, (c - n)/2\rangle \otimes |n\rangle , \tag{11}
\]
where \( S = 0, 1/2, 1, \ldots \) and \( n \) is a non-negative integer such that the quantity \((c - n)/2\) is an integer (if \( S \) is integer) or a half-integer (if \( S \) is half-integer) verifying \( c + 2S \geq n \geq c - 2S \), so that \( n \) is even when \( S \) is integer and \( c \) is even, or when \( S \) is half-integer and \( c \) is odd, and \( n \) is odd when \( S \) is integer and \( c \) is odd, or when \( S \) is half-integer and \( c \) is even. In short, \( n \) is even when \( c + 2S \) is even, and is odd when \( c + 2S \) is odd.

We denote by \( \mathcal{H}^{(S,c)} \) the subspace spanned by the kets (11) for fixed \( S \) and \( c \). Of course, we must have \( c \geq -2S \). For \( c \geq 2S \), the dimension of \( \mathcal{H}^{(S,c)} \) is \( 2S + 1 \) and for \( c < 2S \) the dimension is \((c+2S+1)/2\) if \( c + 2S \) is even and \((c+2S+1)/2\) if \( c + 2S \) is odd.

Depending on the chosen values for \( S \) and \( c \), the physical subspace \( \mathcal{H}^{(S,c)} \) is contained in \( \mathcal{H}_e \) or \( \mathcal{H}_o \), spanned, respectively, by the vectors in (8) and (9). We have
\[
\begin{cases}
\mathcal{H}^{(S,c)} \subset \mathcal{H}_e & \text{if } c + 2S \text{ is even}, \\
\mathcal{H}^{(S,c)} \subset \mathcal{H}_o & \text{if } c + 2S \text{ is odd}.
\end{cases} \tag{12}
\]

We are also interested in the eigenspace of \( S^2 \) associated with the eigenvalue \( S(S+1) \), which we denote by \( \mathcal{H}^{(S)} \). This subspace can be written as the direct sum of the subspaces \( \mathcal{H}_e^{(S)} = \mathcal{H}^{(S)} \cap \mathcal{H}_e \) and \( \mathcal{H}_o^{(S)} = \mathcal{H}^{(S)} \cap \mathcal{H}_o \). In turn, each of these subspaces is the direct sum of physical subspaces \( \mathcal{H}^{(S,c)} \) with \( c \) running over the meaningful values. Since the subspaces \( \mathcal{H}^{(S,c)} \) are left invariant by
the Hamiltonian, it is an easy task to diagonalize the Hamiltonian in $\mathcal{H}_e^{(S)}$ or $\mathcal{H}_o^{(S)}$. We notice also that the subspaces $\mathcal{H}_e^{(S)}$ and $\mathcal{H}_o^{(S)}$ with $S = 0, 1/2, 1, \ldots$ are the irreducible subspaces of the model.

Since the spectrum of $H$ is symmetric in the coupling constant $\lambda$, i.e., the Hamiltonians $H(\lambda)$ and $H(-\lambda)$ have the same eigenvalues, we will assume throughout that $\lambda \geq 0$.

### 3 Trial states

#### 3.1 Coherent states

We want to investigate the ground-state properties and the critical behavior of our model by using suitable trial states. As this is a $su(2) \otimes su(1,1)$ model, i.e., the Hamiltonian is written as a combination of the generators of $su(2) \otimes su(1,1)$, it is natural to consider the $su(2) \otimes su(1,1)$ coherent states as the convenient trial states.

As we saw in Section (2), the irreducible subspaces of the model are the subspaces $\mathcal{H}_e^{(S)}$ and $\mathcal{H}_o^{(S)}$ with $S = 0, 1/2, 1, \ldots$. Therefore, for each $S$, we have two kinds of $su(2) \otimes su(1,1)$ coherent states: one kind corresponds to the subspace $\mathcal{H}_e^{(S)}$ and the other to the subspace $\mathcal{H}_o^{(S)}$. For the subspace $\mathcal{H}_e^{(S)}$, we have the vacuum state $|0_e \rangle = |S, -S \rangle \otimes |0 \rangle$, and for $\mathcal{H}_o^{(S)}$ the vacuum is $|0_o \rangle = |S, -S \rangle \otimes |1 \rangle$. Indeed, we have $S - |S, -S \rangle = 0$ and $a^2 |0 \rangle = 0 = a^2 |1 \rangle$.

The $su(2) \otimes su(1,1)$ coherent states in $\mathcal{H}_e^{(S)}$ are the states

$$|\psi_e \rangle = e^{zS_z + \xi K_+} |0_e \rangle, \quad z, \xi \in \mathbb{C}, \quad |\xi| < 1,$$

and, similarly, for $\mathcal{H}_o^{(S)}$, the associated $su(2) \otimes su(1,1)$ coherent states are

$$|\psi_o \rangle = e^{zS_z + \xi K_+} |0_o \rangle, \quad z, \xi \in \mathbb{C}, \quad |\xi| < 1.$$

The relevant mean values in these states read

$$\frac{\langle \psi_e | S_+ | \psi_e \rangle}{\langle \psi_e | \psi_e \rangle} = S \frac{|z|^2 - 1}{|z|^2 + 1} = \frac{\langle \psi_o | S_+ | \psi_o \rangle}{\langle \psi_o | \psi_o \rangle},$$

$$\frac{\langle \psi_e | S_+ | \psi_e \rangle}{\langle \psi_e | \psi_e \rangle} = 2S \frac{z^*}{|z|^2 + 1} = \frac{\langle \psi_o | S_+ | \psi_o \rangle}{\langle \psi_o | \psi_o \rangle},$$

$$\frac{\langle \psi_e | a^2 | \psi_e \rangle}{\langle \psi_e | \psi_e \rangle} = \frac{|\xi|^2}{1 - |\xi|^2}, \quad \frac{\langle \psi_e | a^2 | \psi_e \rangle}{\langle \psi_e | \psi_e \rangle} = \frac{\xi^*}{1 - |\xi|^2},$$
Since the states $z$ of these mean energies have been already optimized with respect to the phases shown in figure 1, we project these states on the physical subspaces $\mathcal{H}$ and, therefore, the energies of those coherent states are given by

$$\mathcal{E}_e = \frac{\langle \psi_e | H | \psi_e \rangle}{\langle \psi_e | \psi_e \rangle} = \frac{|\xi|^2}{1 - |\xi|^2} + 2S \frac{|z|^2 - 1}{|z|^2 + 1} - 4\lambda S \frac{|z||\xi|}{(|z|^2 + 1)(1 - |\xi|^2)},$$

$$(19)$$

$$\mathcal{E}_o = \frac{\langle \psi_o | H | \psi_o \rangle}{\langle \psi_o | \psi_o \rangle} = \frac{2|\xi|^2 + 1}{1 - |\xi|^2} + 2S \frac{|z|^2 - 1}{|z|^2 + 1} - 12\lambda S \frac{|z||\xi|}{(|z|^2 + 1)(1 - |\xi|^2)}.$$  

$$(20)$$

These mean energies have been already optimized with respect to the phases of $z$ and $\xi$ by choosing $(\arg z - \arg \xi) = \pi$.

Since the states $|\psi_e\rangle$ and $|\psi_o\rangle$ do not belong to the physical subspaces $\mathcal{H}^{(S,e)}$ and $\mathcal{H}^{(S,o)}$, respectively, for fixed eigenvalues $c$ and $c'$ of $C$ such that $\mathcal{H}^{(S,e)} \subset \mathcal{H}^{(S)}_e$ and $\mathcal{H}^{(S,o)} \subset \mathcal{H}^{(S)}_o$, we implement the average conservation of $C$, requiring,

$$\frac{\langle \psi_e | a^{\dagger}a + 2S_z | \psi_e \rangle}{\langle \psi_e | \psi_e \rangle} = \frac{|\xi|^2}{1 - |\xi|^2} + 2S \frac{|z|^2 - 1}{|z|^2 + 1} = c,$$  

$$(21)$$

$$\frac{\langle \psi_o | a^{\dagger}a + 2S_z | \psi_o \rangle}{\langle \psi_o | \psi_o \rangle} = \frac{2|\xi|^2 + 1}{1 - |\xi|^2} + 2S \frac{|z|^2 - 1}{|z|^2 + 1} = c'.$$  

$$(22)$$

In figure 1, we compare the exact ground-state energy with its variational estimate, obtained from (19)-(22), for the cases $S = 5$, $c = 14$ (even case) and $S = 5$, $c = 15$ (odd case). As one can see, the agreement is rather good. The exact ground-state energy is slightly above its variational estimate, but this does not contradict Ritz Theorem, because that is due to some components of the states $|\psi_e\rangle$ and $|\psi_o\rangle$ lying outside the respective physical subspaces. We also find that the agreement is much improved when $c$ is increased. For $c = 100$ and $c = 101$ (with $S = 5$), the lines associated with the even and odd coherent states, respectively, practically coincide with the lines describing the exact results. For $c = 10000$ and $c = 10001$ (with $S = 5$), the associated energy values agree up to the 2nd decimal digit.

### 3.2 Projection on the physical subspaces

Let $c$, $c'$ be fixed eigenvalues of $C$ such that $\mathcal{H}^{(S,e)} \subset \mathcal{H}^{(S)}_e$ and $\mathcal{H}^{(S,o')} \subset \mathcal{H}^{(S)}_o$. In order to improve the description provided by the states $|\psi_e\rangle$ and $|\psi_o\rangle$, as shown in figure 1, we project these states on the physical subspaces $\mathcal{H}^{(S,e)}$ and

$$\frac{(\psi_o | a^{\dagger}a | \psi_o)}{(\psi_o | \psi_o)} = \frac{2|\xi|^2 + 1}{1 - |\xi|^2}, \quad \frac{(\psi_o | a^{\dagger}a | \psi_o)}{(\psi_o | \psi_o)} = \frac{3\xi^*}{1 - |\xi|^2},$$

and, therefore, the energies of those coherent states are given by

$$\mathcal{E}_e = \frac{\langle \psi_e | H | \psi_e \rangle}{\langle \psi_e | \psi_e \rangle} = \frac{|\xi|^2}{1 - |\xi|^2} + 2S \frac{|z|^2 - 1}{|z|^2 + 1} - 4\lambda S \frac{|z||\xi|}{(|z|^2 + 1)(1 - |\xi|^2)},$$

$$(18)$$

$$\mathcal{E}_o = \frac{\langle \psi_o | H | \psi_o \rangle}{\langle \psi_o | \psi_o \rangle} = \frac{2|\xi|^2 + 1}{1 - |\xi|^2} + 2S \frac{|z|^2 - 1}{|z|^2 + 1} - 12\lambda S \frac{|z||\xi|}{(|z|^2 + 1)(1 - |\xi|^2)}.$$  

$$(20)$$

In figure 1, we compare the exact ground-state energy with its variational estimate, obtained from (19)-(22), for the cases $S = 5$, $c = 14$ (even case) and $S = 5$, $c = 15$ (odd case). As one can see, the agreement is rather good. The exact ground-state energy is slightly above its variational estimate, but this does not contradict Ritz Theorem, because that is due to some components of the states $|\psi_e\rangle$ and $|\psi_o\rangle$ lying outside the respective physical subspaces. We also find that the agreement is much improved when $c$ is increased. For $c = 100$ and $c = 101$ (with $S = 5$), the lines associated with the even and odd coherent states, respectively, practically coincide with the lines describing the exact results. For $c = 10000$ and $c = 10001$ (with $S = 5$), the associated energy values agree up to the 2nd decimal digit.
Figure 1: Lowest energy in the physical subspaces as function of the coupling constant $\lambda$. The results represented in (a) correspond to the even physical subspace, and those in (b) to the odd one. In both figures, the thick line represents the exact results and the thin line the minimum energy of the coherent states ($|\psi_e\rangle$ in (a) and $|\psi_o\rangle$ in (b)). The minimum energy of the projected coherent state ($|\psi_{e,p}\rangle$ in (a) and $|\psi_{o,p}\rangle$ in (b)) is represented by a line which coincides with the exact result.

$\mathcal{H}^{(S,c')}$, respectively. The projected states, apart from multiplicative factors, read, respectively,

$$|\psi_{e,p}\rangle = \sum_{n=n_0}^{S+\frac{s}{2}} \frac{\rho^n}{(S+\frac{s}{2}-n)!n!} S_+^{S+\frac{s}{2}-n} K_+ |0_e\rangle,$$

$$|\psi_{o,p}\rangle = \sum_{n=n'_0}^{S+\frac{s'-1}{2}} \frac{\rho^n}{(S+\frac{s'-1}{2}-n)!n!} S_+^{S+\frac{s'-1}{2}-n} K_+ |0_o\rangle,$$

where $\rho = \xi z$, $n_0 = \text{Max}(0, \frac{s}{2} - S)$ and $n'_0 = \text{Max}(0, \frac{s'-1}{2} - S)$. We have

$$(\psi_{e,p}|\psi_{e,p}) = \sum_{n=n_0}^{S+\frac{s}{2}} \left( \frac{2S}{S+\frac{s}{2}-n} \right) \frac{(2n)!}{(2n!n!)^2} |\rho|^{2n},$$

$$(\psi_{o,p}|\psi_{o,p}) = \sum_{n=n'_0}^{S+\frac{s'-1}{2}} \left( \frac{2S}{S+\frac{s'-1}{2}-n} \right) \frac{(2n+1)!}{(2n!n!)^2} |\rho|^{2n},$$
where \( \binom{N}{p} = \frac{N!}{p!(N-p)!} \). The mean energies of those states read

\[
\mathcal{E}_{e,p} = \frac{\langle \psi_{e,p} | H | \psi_{e,p} \rangle}{\langle \psi_{e,p} | \psi_{e,p} \rangle} = c - \frac{2\lambda}{\langle \psi_{e,p} | \psi_{e,p} \rangle} \sum_{n=n_0}^{S+\frac{c}{2}-1} \frac{(2S)!(2n+1)!|\rho|^{2n+1}}{(S - \frac{c}{2} + n)!(S + \frac{c}{2} - n - 1)!\left(2^n n!\right)^2},
\]

\( (27) \)

\[
\mathcal{E}_{o,p} = \frac{\langle \psi_{o,p} | H | \psi_{o,p} \rangle}{\langle \psi_{o,p} | \psi_{o,p} \rangle} = c' - \frac{2\lambda}{\langle \psi_{o,p} | \psi_{o,p} \rangle} \sum_{n=n'_0}^{S+\frac{c'-1}{2}-1} \frac{(2S)!(2n+1)!(2n+3)!|\rho|^{2n+1}}{(S - \frac{c'-1}{2} + n)!(S + \frac{c'-1}{2} - n - 1)!\left(2^n n!\right)^2},
\]

\( (28) \)

where optimization with respect to \( \arg \rho \) has already been performed. Since these energies are finite order polynomials in \( \rho \), they can be easily treated by computational methods. Their minimum values, as function of \( \lambda \), are represented in figure 1, where they appear superimposed on the exact results, showing an excellent agreement which still improves when \( c \) increases. For example, for \( c = 10000 \) and \( c = 10001 \) (with \( S = 5 \)), the associated energy values agree up to the 8th decimal digit which is a spectacular agreement. The performance of the projected coherent states \( |\psi_{e,p}\rangle \) and \( |\psi_{o,p}\rangle \) is superior to the one of the coherent states \( |\psi_e\rangle \) and \( |\psi_o\rangle \), respectively, as expected.

4 The completed two-photon JC model

Up to now, we concentrated on the physical subspaces \( \mathcal{H}^{(S,c)} \), i.e., on the states characterized by specific eigenvalues \( S(S+1) \) and \( c \) of the constants of motion \( S^2 \) and \( C \), respectively. When the constant of motion \( C \) is not explicitly specified, the relevant Hilbert subspace of the model is the whole eigenspace of \( S^2 \), \( \mathcal{H}^{(S)} \). We find, with some surprise, that, in \( \mathcal{H}^{(S)} \), the energy has a lower bound only if \( \lambda \leq 1/(2S) \). For \( \lambda > 1/(2S) \), it is possible to find eigenenergies as low as one wishes, so that no ground-state exists. The Hamiltonian is unbounded from below. In this sense, we say that the model is incomplete for \( \lambda > 1/(2S) \). This can be readily seen by minimizing \( \mathcal{E}_e \) (equation (19)) with respect to \( z \) and \( \xi \). We can also observe this situation by studying the dependence on \( c \) of the exact ground-state energy of the physical subspace \( \mathcal{H}^{(S,c)} \), which we denote by \( E_0^{(S,c)} \). As may be seen from figure 2(a), for \( \lambda \leq 1/(2S) \), the energy \( E_0^{(S,c)} \) increases with \( c \). However, this does not happen if \( \lambda > 1/(2S) \). In this case, \( E_0^{(S,c)} \) decreases indefinitely when \( c \) increases after a certain critical value, as is
shown in figure 2(b). This effect illustrates dramatically the phase transition from the normal to the so-called super-radiant phase.

The global minimum of $E_e$ is seen to be $(E_e)_{\text{min}} = -2S$ for every $\lambda \leq 1/(2S)$ and this agrees with the exact results. In fact, for each $\lambda \leq 1/(2S)$, the exact ground-state energy is $E_0^{(S,c)} = -2S$ which occurs for $c = -2S$. As for the energy $E_o$, its global minimum is $(E_o)_{\text{min}} = -2S + 1$ for every $\lambda \leq 1/(2S)$ and it has no minimum if $\lambda > 1/(2S)$.

Figure 2: $E_0^{(S,c)}$ as function of $c$, for $S = 5$ and fixed $\lambda$. In (a), $\lambda = 0.1$ (normal phase). In (b), $\lambda = 0.1001$ (super-radiant phase).

We stabilize the model by adding to the Hamiltonian (3) a term quadratic in $C$. We consider then the stabilized Hamiltonian

$$H_\epsilon = a^\dagger a + 2S_z + \epsilon(a^\dagger a + 2S_z)^2 + \lambda \left( S_+ a^2 + S_- a^4 \right),$$

(29)

where $\epsilon$ is a positive real parameter which is small enough so that the desirable physical features of the original model are not destroyed. In the sequel, we will refer to $H_\epsilon$ as the completed two-photon Jaynes-Cummings (c2JC) model.

The original Hamiltonian and the new one have the same eigenstates, their eigenvalues being displaced by $\epsilon c^2$. The conclusions we draw for the stabilized Hamiltonian have, hence, a counterpart in the original one.

In particular, figure 3 shows that the new Hamiltonian is indeed stabilized. In figure 3(a), $E_0^{(S,c)}$ is represented as function of $c$, for $\lambda$ above the critical point, as figure 2(b) does for the original model. Now, it is seen that, indeed, the energy has a minimum. Notice also that the minimum occurs for a rather
high value of $c$, even though the coupling constant is not far from the critical value. Below the critical value, the minimum energy occurs for $c = -2S$. These facts are clearly depicted in figure 4.

An important aspect of the 2JC model is the phase transition for $\lambda = 1/(2S)$. In figure 3(b), we represent the exact ground-state energy in $\mathcal{H}^{(S)}$ by a line which practically coincides with the thick line, and, as may be seen, the phase-transition is also clearly exhibited for the c2JC model, as long as we take small enough values of $\epsilon$. For $\epsilon = 0.005$, the critical value is $\lambda_c = 0.13845$, so that the stabilizing term pushes the critical point to a somewhat higher value of $\lambda$. When $\epsilon$ decreases to 0, the critical value of $\lambda$ approaches $\lambda = 1/(2S)$.

It may be interesting to investigate to which of the invariant subspaces $\mathcal{H}^{(S)}_e$ and $\mathcal{H}^{(S)}_o$ the exact ground-state of $\mathcal{H}^{(S)}$ actually belongs, for each $\lambda$ value. Figure 4 answers this question. Since each vertical step in the curve of $c_{\text{min}}$ (after the transition) corresponds to adding one unit to the value of $c_{\text{min}}$, each step...
Figure 4: Properties of the c2JC model: The value $c_{\text{min}}$ of $c$ which minimizes the energy $E_0(S,c)$ in the whole $\mathcal{H}^{(S)}$, as function of $\lambda$. The discrete nature of $c$ is clearly exhibited. After the critical value, each vertical step corresponds to adding one unit to the value of $c_{\text{min}}$.

corresponds also to a change in the invariant subspace to which the ground-state of $\mathcal{H}^{(S)}$ belongs, as may be seen from (12). When $\lambda$ is gradually increased, the properties of the system change discontinuously because $c_{\text{min}}$ is quantized. However, the properties of the super-radiant phase, as fig. 3(a) shows, depend on $c$ almost continuously, no discontinuous behavior being noticeable. The actual ground-state in $\mathcal{H}^{(S)}$ switches permanently from $\mathcal{H}^{(S)}_e$ to $\mathcal{H}^{(S)}_o$, or vice versa, the ground-state energies being practically the same in $\mathcal{H}^{(S)}_e$ and $\mathcal{H}^{(S)}_o$. The same applies to the average numbers of photons in the relative ground-states of $\mathcal{H}^{(S)}_e$ and $\mathcal{H}^{(S)}_o$ which are very close to each other.

In figure 3(b), the exact ground-state energies in $\mathcal{H}^{(S)}_e$ and $\mathcal{H}^{(S)}_o$ are represented. These are the full and dashed thick lines, respectively. After the critical point, these two lines practically coincide. The absolute ground-state energy in $\mathcal{H}^{(S)}$ is clearly the lowest of the ground-state energies in $\mathcal{H}^{(S)}_e$ and in $\mathcal{H}^{(S)}_o$. Before the transition, the exact ground-state energy in $\mathcal{H}^{(S)}$ occurs for $c = -2S$ (figure 4) which corresponds to the even subspace. Thus the exact ground-state energies in $\mathcal{H}^{(S)}_e$ and $\mathcal{H}^{(S)}_o$ coincide before the transition. For $\mathcal{H}^{(S)}_o$, the ground-state energy before the phase transition occurs for $c = -2S + 1$ which corresponds to a slightly higher energy.

In figure 3(b), we also compare the minimum energies of the coherent states $|\psi_e\rangle$ and $|\psi_o\rangle$ with the exact ground-state energies in $\mathcal{H}^{(S)}_e$ and $\mathcal{H}^{(S)}_o$, respectively. As we see, in both cases, the agreement is excellent before the transition and reasonable after it. This happens because, within the physical subspaces $\mathcal{H}^{(S,c)}$, the performances of $|\psi_e\rangle$ and $|\psi_o\rangle$ are good, as shown in figure 5, if $c$ is
not too big. However, they become gradually worse when $c$ increases, and a
sizeable discrepancy is already noticeable in figure 6 for $c = 300$ and $c = 301$.
This discrepancy is due to the fluctuations in $a^\dagger a$ and $S_z$. On the other hand,
after the transition, the exact ground-states in $\mathcal{H}_e^{(S)}$ and $\mathcal{H}_o^{(S)}$ occur for rather
large values of $c$, for which the performances of $|\psi_e\rangle$ and $|\psi_o\rangle$ are reasonable
but no longer good. This deficiency of the states $|\psi_e\rangle$ and $|\psi_o\rangle$ is corrected in
the projected states $|\psi_{e,p}\rangle$ and $|\psi_{o,p}\rangle$. In figure 3(b), the lines representing the
estimates of the ground-state energies of $\mathcal{H}_e^{(S)}$ and $\mathcal{H}_o^{(S)}$ based on the projected
states $|\psi_{e,p}\rangle$ and $|\psi_{o,p}\rangle$, respectively, practically coincide with the associated
exact results. In fact, the performance of the projected states in the c2JC
model is always excellent no matter how big is the value of $c$. This high per-
formance of $|\psi_{e,p}\rangle$ and $|\psi_{o,p}\rangle$ is due to the eigenstates of the stabilized model
being the same as those of the original one.

In figure 7(a), we plot the ratio between the correlation energies $(E_f - E_p)$,
estimated in the projection method, and $(E_f - E_{ex})$, the exact one, for each
of the invariant subspaces $\mathcal{H}_e^{(S)}$ and $\mathcal{H}_o^{(S)}$. In both cases, this ratio is every-
where equal to 1 except in the immediate vicinity of the critical points where
it approaches zero. This confirms the excellent performance of the projected
states.

In figure 7(b), the average number of photons, regarded as an order parameter, is represented for the ground-states of $H^{(S)}_e$ and $H^{(S)}_o$. Before the transition, we have $\langle a^\dagger a \rangle = 0$ for the even case and $\langle a^\dagger a \rangle = 1$ for the odd one. After it, $\langle a^\dagger a \rangle$ increases indefinitely with $\lambda$. The behavior predicted by the coherent states $|\psi_o\rangle$ and $|\psi_e\rangle$, such as the occurrence of the super-radiant phase transition, is qualitatively correct but not quantitatively satisfactory. We remark that the performance of $|\psi_o\rangle$ is superior to the performance of $|\psi_e\rangle$, which may be connected with the fact that, essentially, $|\psi_o\rangle = a^\dagger |\psi_e\rangle$.

The behavior of $\langle S_z \rangle$ in the ground-states of $H^{(S)}_e$ and $H^{(S)}_o$ is shown in figure 8(a). For both cases, before the transition, $\langle S_z \rangle = -5$, i.e., all spins are down. After the transition, when $\lambda$ increases, $\langle S_z \rangle$ approaches zero asymptotically. Again, the performance of the coherent states $|\psi_e\rangle$ and $|\psi_o\rangle$ is excellent before the transition and reasonable after it. In turn, the performance of the projected states is always remarkable.

In figure 8(b), the variance of $S_z$ in the ground-states of $H^{(S)}_e$ and $H^{(S)}_o$ is represented. Before the transition, $\Delta S_z = 0$ in both cases, and after it $\Delta S_z$ approaches asymptotically 2.5 when $\lambda$ is increased. In both figures 8(a) and 8(b), before the transition, there is no difference between the even and the odd cases. This was to be expected, since the difference between those subspaces
Figure 7: Properties of the c2JC model: correlation energy and order parameter. (a) Ratio between the correlation energy estimated by the projection method, $E_f - E_p$, and the exact correlation energy $E_f - E_{ex}$, for $\mathcal{H}_e(S)$ (full line) and for $\mathcal{H}_o(S)$ (dashed line). Here, $E_f$ is the ground-state energy estimated by the associated coherent state (| $\psi_e$ > for $\mathcal{H}_e(S)$ and | $\psi_o$ > for $\mathcal{H}_o(S)$), $E_p$ is the ground-state energy estimated by the associated projected coherent state (| $\psi_{e,p}$ > for $\mathcal{H}_e(S)$ and | $\psi_{o,p}$ > for $\mathcal{H}_o(S)$), $E_{ex}$ is the exact ground-state energy. (b) Average of $a^\dagger a$, regarded as order parameter, in the ground-states of $\mathcal{H}_e(S)$ and $\mathcal{H}_o(S)$. The number of photons in the exact ground-states of $\mathcal{H}_e(S)$ and $\mathcal{H}_o(S)$ are represented by the thick full and thick dashed lines, respectively. The thin full and thin dashed lines refer to the coherent states | $\psi_e$ > and | $\psi_o$ >, respectively. The lines corresponding to the projected coherent states are superimposed on the corresponding exact results. The exact ground-state in $\mathcal{H}(S)$ is represented by a line which practically coincides with the thick full line.
Two-photon Jaynes-Cummings model

5 Conclusions

The ground-state properties of the two-photon Jaynes-Cummings model were investigated using \( su(2) \otimes su(1, 1) \) coherent states, in the framework of conventional mean-field technics. We found that variational results so obtained compare favorably with exact results. Furthermore, our results are much improved if the constant of motion of the model is implemented exactly, by projecting the coherent states on the physical subspaces, instead of being implemented only in the average.

We have remarked that this model lacks a ground-state, i.e., the spectrum of its Hamiltonian has the undesirable feature of being unbounded from below. In order to circumvent this drawback a stabilized version of the model was proposed and its ground-state properties were investigated using again \( su(2) \otimes su(1, 1) \) coherent states. Similarly to the original model, the coherent states provide a good description of the new model, specially the projected states. Also, the ground-state which is absent from the original model is now reasonably well described by the coherent states, the performance of the pro-
jected states being excellent. Some quantities of interest, such as $\langle a^\dagger a \rangle$, $\langle S_z \rangle$ and $\Delta S_z$, were calculated for the ground-state and compared with the associated estimates based on the coherent states including the projected ones, confirming that the investigated coherent states provide a good description of the model. Those quantities provide useful information concerning the superradiant phase transition. In fact, the phase transition is clearly exhibited in the graphical representation of the behavior of those quantities.

The present work is close in spirit to the article [24] by the same authors. In [24], the $su(2) \otimes su(1, 1)$ coherent states were used to investigate the ground-state properties of the Buck-Sukumar model by the use of conventional mean-field technics, and, as here, they were seen to provide a good description of the model. The conclusions obtained in both works support the following conjecture: Let $H$ be the Hamiltonian of a certain model described by a Lie algebra $\mathcal{G}$, i.e., $H$ is a combination of elements of $\mathcal{G}$. Then the coherent states of $\mathcal{G}$ are good choices to describe that model, in the sense that if $E_0$ is the exact ground-state energy of $H$ in some physical subspace and $E_{\text{min}}$ is the ground-state energy, in the same physical subspace, estimated by the coherent states of $\mathcal{G}$ by using conventional mean-field technics, then $E_{\text{min}} \approx E_0$.

References


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