Symmetry Breaking and Exact Solutions of the Hyperbolic Heat Equation with Variable Medium Properties

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Abstract

We classify from Lie symmetry standpoint the hyperbolic heat equation with temperature-dependent medium properties. We prove that the hyperbolic heat equation may admit a two-, three-, four- or infinite-dimensional symmetry Lie algebra depending on the functional form of density, thermal conductivity, and specific heat. The last possibility corresponds to exactly linearizable hyperbolic heat equation. The symmetry structure of the hyperbolic heat equation is used to optimally classify similarity solutions. Also we employ group foliation technique to obtain separable solutions of the hyperbolic heat equation.

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1 Introduction

The classical heat equation asserts that heat propagates at an infinite speed. Although reasonable in several engineering applications, experimental results indicate that this assumption breaks down if the medium is very thin, temperature is very low or propagation time is extremely short. Indeed, empirical findings suggest that under certain conditions, heat propagates at a finite speed and behaves like a wave. This duality of heat propagation was already noticed in the early 40’s: Peshkov[18] showed on the monitor of an oscilloscope a pulse energy source propagating at 19m/s in helium at 1K. Peshkov’s thermal wave
was later explained by Bak [1] using phonon and roton interaction theory. Another (indirect) proof of the existence of thermal wave can be found in the study of exothermic catalytic reactions in crystals: Cusumano and Low [8] noticed an abnormal rise of temperature in the reaction of oxygen with silicon dioxide in presence of nitrate. In a time period of about $10^{-3}$ s, they reported a temperature increase from 2000K to 3000K whereas predictions using the classical heat equation are in the range 2K to 250 K [13, 19]. Using a hyperbolic heat model, Chan et al.[4] predicted a temperature rise from 1100K to 2500K. This results seems to indicate that heat conduction in short times is a wave phenomenon. Thermal waves have several applications amongst which we may cite cryogenic engineering, super-conductors, manufacturing of crystals, measurement of impurities in anisotropic materials, nuclear engineering and seismology.

There are several derivations of models that account for the wave nature of heat conduction [1, 5, 6, 7, 20, 22]. The starting point for the derivation of these models is generally quantum mechanics (phonon theory), statistical mechanics (Boltzmann equation, random walk) or a modification of Fourier’s Law. The latter approach was initiated by Vernotte [20] and Cattaneo [3] who replaced the classical Fourier Law by

$$q + \tau \frac{\partial q}{\partial t} = -K \nabla T,$$

where $\nabla$ is the spatial gradient operator, $q$ is heat flux, $K$ thermal conductivity, $T$ is temperature and $\tau = \alpha_0/c_0^2 = K_0/(\rho_0 C_\rho c_0^2)$ is relaxation time, and $\alpha_0$, $c_0$, $\rho_0$, $K_0$ and $C_\rho$ are respectively reference thermal diffusivity, propagation speed, density, thermal conductivity and specific heat. Upon coupling Eq. (1) with the continuity equation

$$-\nabla . q = \frac{\partial (\rho C_p T)}{\partial t},$$

where $C_p$ is the specific heat, we obtain the hyperbolic heat equation

$$\tau \frac{\partial^2 (\rho C_p T)}{\partial t^2} - \nabla . (K \nabla T) + \frac{\partial (\rho C_p T)}{\partial t} = 0.$$  

In the sequel, we assume that we have a single spatial variable $x$ and that medium properties are temperature-dependent. We may introduce the following non-dimensional variables [16]

$$\bar{t} = \frac{c_0^2 t}{2\alpha_0}, \quad \bar{x} = \frac{c_0 x}{2\alpha_0}, \quad u = \frac{T - T_f}{T_i - T_f},$$

where $T_i$ and $T_f$ are reference initial and final temperatures. In terms of the new variable and dropping the bars, Eq. (3) in one spatial variable is
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\[ (f(u)u_t)_t - (g(u)u_x)_x + 2f(u)u_t = 0, \quad (4) \]

where indices stand for partial differentiations, \( f(u) = \rho(T)C_p(T)/\rho_0C_p0 \) and \( g(u) = K(T)/K_0 \).

Our aim in this paper is threefold: find specifications of the the functions \( f \) and \( g \) that render Eq. (4) maximally symmetric, investigate invariant solutions of maximally symmetric models, and obtain separable solutions via group foliation technique. To place this work in its proper perspective, it worth noting that symmetry analysis of second-order hyperbolic equations was initiated by Sophus Lie [12] who completely classified from a symmetry standpoint linear equations. Subsequent works were concerned with classification of specific classes of nonlinear hyperbolic equations (see [10, 11] for a review and new results). Except for the paper by Pakdemirli and Şahin [16] dealing with approximate symmetries of Eq. (4), there has not been to the best of our knowledge a paper dealing with exact symmetries and invariant solutions of hyperbolic equations equivalent to Eq. (4).

We have organized this paper as follows. There are five sections of which this introduction constitutes the first. In the second section, we perform a complete symmetry classification of Eq. (4). Section 3 is concerned with the investigation of invariants solutions of Eq. (4). Section 4 deals with separable solutions of Eq. (4). We summarize our findings in Section 5.

\[ \text{2} \quad \text{Complete symmetry classification of the hyperbolic heat equation} \]

Here we study Lie point symmetries [2, 14, 15] of the hyperbolic heat equation (4). Note that the direct study of the symmetries of Eq. (4) leads to determining equations that are difficult to analyze. In order to overcome this problem, we shall start by invertibly mapping Eq. (4) to a simpler equation.

Multiply Eq. (4) through by \( e^{2t} \) and rewrite the resulting equation as

\[ (e^{2t}f(u)u_t)_t - e^{2t}(g(u)u_x)_x = 0. \quad (5) \]

Equation (5) suggests the change of variables

\[ \tau = \frac{1}{2} e^{-2t}, \quad v = \int^u f(\theta) \, d\theta. \quad (6) \]

In the new variables (6), Eq. (4) reads

\[ 4\tau^2 v_{\tau\tau} - (h(v)v_x)_x = 0, \quad (7) \]
where
\[ h(w) = \frac{g(u)}{f(u)}. \] (8)

A vector field
\[ X = \xi^1(\tau, x, v) \frac{\partial}{\partial \tau} + \xi^2(\tau, x, v) \frac{\partial}{\partial x} + \phi(\tau, x, v) \frac{\partial}{\partial v}, \] (9)
is a point symmetry [2, 14, 15] of Eq. (4) if and only if
\[ X^{[2]}(4\tau^2 v_{\tau \tau} - (h(v)v_x)_x)|_{(\tau)} = 0, \] (10)
where
\[ X^{[2]} = X + \phi^{x\tau} \frac{\partial}{\partial v_x} + \phi^{x\tau \tau} \frac{\partial}{\partial v_{\tau \tau}} + \phi^{x x} \frac{\partial}{\partial v_{x x}}, \] (11)
\[ \phi^{\nu \nu} = D_{\tau}(\phi) - v_{\tau} D_{x}(\xi^1) - v_{x} D_{x}(\xi^2), \] (12)
\[ \phi^{\nu x} = D_{x}(\phi) - v_{\tau} D_{x}(\xi^1) - v_{x} D_{x}(\xi^2), \] (13)
\[ \phi^{x \nu \tau} = D_{\tau}(\phi^{x \nu}) - v_{\tau \tau} D_{\tau}(\xi^1) - v_{\tau x} D_{x}(\xi^2), \] (14)
\[ \phi^{x x} = D_{x}(\phi^{x x}) - v_{\tau x} D_{x}(\xi^1) - v_{x x} D_{x}(\xi^2), \] (15)
where the operator of total differentiation \( D_{\lambda} \) is defined by
\[ D_{\lambda} = \frac{\partial}{\partial \lambda} + v_{\lambda} \frac{\partial}{\partial v} + v_{\lambda \alpha} \frac{\partial}{\partial v_{\alpha}} + \cdots \] (16)

Since the symmetry coefficients \( \xi^1, \xi^2 \) and \( \phi \) are independent of the derivatives of \( v \) and Eq. (7) is polynomial in these derivatives, like coefficients on both sides of Eq. (7) may be equated to yield an overdetermined system of linear partial differential equations (PDEs). After some simplifications, the determining equations for symmetries read:

\[ \xi^1 = \xi^1(\tau), \quad \xi^2 = \xi(x), \quad \phi = \left( \frac{1}{2} \xi^1_{\tau} + a(x) \right) v + b(\tau, x), \] (17)
\[ 4(\xi^1 - \tau \xi^2_{\tau}) h - 2\tau b h' - \tau(2a + \xi^1_{\tau}) v h' = 0, \] (18)
\[ (4\xi^1 - 5\tau \xi^1_{\tau} + 4\tau \xi^2_{x} - 2\tau a) h' - 2\tau b h'' - \tau(2a + \xi^1_{\tau}) v h'' = 0, \] (19)
\[ h b_{\tau \tau} + a'' v h - 2\tau^2 v \xi^1_{\tau \tau \tau} - 4\tau^2 b_{\tau \tau} = 0, \] (20)
\[ (\xi^2_{x} - 2a') h - 2b_x h' - 2a' v h' = 0. \] (21)

Let \( \Gamma \) be the vector space spanned by \( h, h' \) and \( v h' \) i.e \( \Gamma = < h, h', v h' > \). Equation (21) prompts the consideration of the cases \( \dim \Gamma = 3, \dim \Gamma = 2 \) and \( \dim \Gamma = 1 \).

**Case 1:** \( \dim \Gamma = 3 \). Solving Eq. (18) and Eq. (21), we obtain
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\[ \xi^1 = -2c_1 \tau, \quad \xi^2 = c_2, \quad a = c_1, \quad b = 0 \quad \phi = 0, \quad (22) \]

where \( c_1 \) and \( c_2 \) are arbitrary constants. It can be verified that the remaining equations \textit{viz.} Eq. (19) and Eq. (20) are identically satisfied. Thus in this case, the symmetry Lie algebra is spanned by the vector field

\[ X_1 = \tau \frac{\partial}{\partial \tau}, \quad X_2 = \frac{\partial}{\partial x}. \quad (23) \]

**Case 2:** \( \dim \Gamma = 2 \). In this case \( h = \alpha h' + \beta vh' \), where \( \alpha \) and \( \beta \) are constants such that \( (\alpha, \beta) \neq (0, 0) \). Hence

\[ \frac{h'}{h} = \frac{1}{\alpha + \beta v}. \quad (24) \]

The solution of Eq. (24) yields

\[ h = c e^{mv} \text{ or } h = c (v + d)^m \quad (25) \]

where \( m = 1/\alpha, \beta = 0 \) in Eq. (25a) and \( m = 1/\beta, \quad d = \alpha/\beta, \beta \neq 0 \) in Eq. (25b), and \( c \) is a constant.

If \( h = c e^{mv}, m \neq 0 \), the solution of Eqs. (17)-(21) is

\[ \xi^1 = -2c_1 \tau, \quad \xi^2 = mc_2 x + c_3, \quad \phi = c_2, \quad (26) \]

where \( c_1, c_2, \) and \( c_3 \) are constants. Thus the symmetry Lie algebra is spanned by the operators

\[ X_1, \quad X_2, \quad X_3 = mx \frac{\partial}{\partial x} + 2 \frac{\partial}{\partial v}. \quad (27) \]

If \( h = c (v + d)^m \) and \( m \neq 0, -4/3 \), the symmetry coefficients are

\[ \xi^1 = c_1 \tau, \quad \xi^2 = \frac{m}{4} (2c_2 + c_1)x + c_3, \quad \phi = \left( c_2 + \frac{1}{2} c_8 \right) (v + d), \quad (28) \]

where \( c_1, c_2, \) and \( c_3 \) are constants. Hence the symmetry Lie algebra is generated by

\[ X_1, \quad X_2, \quad X'_3 = mx \frac{\partial}{\partial x} + 2(v + d) \frac{\partial}{\partial v}. \quad (29) \]

If \( h = (c(v + d)^{-4/3} \), the solution of the determining equations for symmetries is

\[ \xi^1 = c_1 \tau, \quad \xi^2 = -\frac{1}{3} c_4 x^2 - \frac{1}{3} c_3 x + c_2, \quad \phi = \left( c_4 x + \frac{1}{2} c_3 \right) v + c_4 dx + \frac{d}{2} c_3, \quad (30) \]
where \( c_1 \) to \( c_4 \) are constants, and the generators of the symmetry Lie algebra are

\[
X_1, \quad X_2, \quad X'_3(m = -4/3), \quad X_4 = x^2 \frac{\partial}{\partial x} - 3(v + d)x \frac{\partial}{\partial v}.
\]  (31)

**Case 3:** \( \dim \Gamma = 1 \). This case corresponds to \( h = A = \text{constant} \). Solving Eqs (18)-(21), we obtain

\[
\xi^1 = c_1 \tau, \quad \xi^2 = c_2, \quad \phi = c_3 u + b(\tau, x),
\]  (32)

where \( c_1 \) to \( c_3 \) are constants and \( b \) is a solution of the equation

\[
4\tau^2 b_{\tau\tau} - Ab_{xx} = 0.
\]  (33)

Hence the symmetry Lie algebra is generated by the operators

\[
X_1, \quad X_2, \quad X'_3 = v \frac{\partial}{\partial v}, \quad X_b = b(\tau, x) \frac{\partial}{\partial v},
\]  (34)

where \( b \) solves Eq. (33). Having completed the symmetry classification of Eq. (7), we are left with transferring the results to the hyperbolic heat equation (4). In this process the key ingredient is Eq. (6) from which we infer that

\[
\frac{\partial}{\partial \tau} = -e^{2t} \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial v} = f(u) \frac{\partial}{\partial u}.
\]  (35)

Using the symmetry classification of Eq. (7), Eq. (6) and its byproduct Eq. (35), we obtain the following theorem.

**Theorem 2.1** There are five classes of maximally symmetric hyperbolic heat equation (4) given by the following specifications of \( f \) and \( g \).

**(i)** \( g/f \) is an arbitrary functions of \( u \), and the symmetry Lie algebra is spanned by the operators

\[
Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial x}.
\]  (36)

**(ii)** \( g(u) = cf(u) \exp \left[ m \int^u f(\theta)d\theta \right] \), \( m \neq 0 \), and the symmetry Lie algebra is generated by

\[
Y_1, \quad Y_2, \quad Y_2 = mx \frac{\partial}{\partial x} + 2f(u) \frac{\partial}{\partial u}.
\]  (37)

**(iii)** \( g(u) = cf(u) \left( \int^u f(\theta)d\theta + d \right)^m \), \( m \neq 0, -4/3 \), and the generators of the symmetry Lie algebra are

\[
Y_1, \quad Y_2, \quad Y'_3 = mx \frac{\partial}{\partial x} + 2f(u) \left( \int^u f(\theta)d\theta + d \right) \frac{\partial}{\partial u}.
\]  (38)
(iv) \( g(u) = cf(u)\left(\int u f(\theta)d\theta + d\right)^{-4/3} \), and the symmetry Lie algebra is spanned by

\[ Y_1, \: Y_2, \: Y'_3(m = -4/3), \: Y_4 = x^2 \frac{\partial}{\partial x} - 3f(u) \left(\int u f(\theta)d\theta + d\right) x \frac{\partial}{\partial u}. \] (39)

(v) \( g(u) = A f(u), \: A = \text{const.} \), and the symmetry Lie algebra is generated by

\[ Y_1, \: Y_2, \: Y''_3 = f(u) \left(\int u f(\theta)d\theta\right) \frac{\partial}{\partial u}, \: Y_b = b \left(-\frac{1}{2} e^{-2t}, x\right) f(u) \frac{\partial}{\partial u}. \] (40)

where \( b(\tau, x) \) solve the PDE \( 4\tau^2 b_{\tau\tau} - Ab_{xx} = 0 \).

A byproduct of Theorem 2.1 is the following.

**Corollary 2.2** The hyperbolic heat equation (4) is exactly linearizable if and only if \( g \propto f \).

**Proof.** The necessity is straightforward. Indeed, if Eq. (4) is exactly linearizable, it must admit and infinite-dimensional symmetry Lie algebra [2]. From the symmetry classification of Eq. (4) it may be inferred that Eq. (4) possesses and infinite-dimensional symmetry Lie algebra provided \( g \propto f \). Conversely, if \( g \propto f \), say, \( g = Af \), the invertible transformation Eq. (6) maps Eq. (4) to the linear PDE \( \tau^2 v_{\tau\tau} - Av_{xx} = 0 \).

3 Invariant solutions of the hyperbolic heat equation

In order to classify, invariant solutions, we need to obtain nonsimilar subalgebras of the symmetry Lie algebras i.e. solve the optimal system problem [14, 15]. Save for the linear case, our symmetry Lie algebras are at most four-dimensional. Subalgebras of Lie algebras up to dimension four where completely classify by Patera and Winternitz [17]. Thus, for our purpose, we just have to identify in their classification Lie algebras corresponding to our symmetry algebras. This is done by computing the Lie brackets of the symmetry operators. Once optimal systems are obtained, the ansatz for an invariant solution \( u = I(\tau, x) \) corresponding to an element \( X \) of an optimal system is obtained by solving the first-order PDE \( X[u - I(\tau, x)]|_{u=I} = 0 \).
3.1 $f/g$ is an arbitrary function of $u$

In this case the symmetry Lie algebra corresponds to the abelian two-dimensional Lie algebra $2A_1$. Thus an optimal system of 1D subalgebras of the 2D symmetry Lie is $\{Y_1, Y_2 + cY_1\}$, where $c$ is a real number. Solutions invariant under $Y_1$ are steady solutions of Eq.(4) given by

$$\int^u g(r)dr = C_1x + C_2,$$

where $C_1$ and $C_2$ are integration constants. Solutions invariant under $Y_2 + cY_1$ are the so-called traveling wave solutions of Eq. (4). They are solutions of the ordinary differential equation (ODE)

$$(f(w)c^2 - g(w))w'' + (c^2f'(w) - g'(w))w'^2 + 2cf(w)w' = 0. \tag{42}$$

Equation (42) is invariant under translation of the independent variable $x - ct$. Thus its order can be further reduced by one by introducing the new dependent variable $y = w'$ and using $w$ as the new independent variable (note that Eq. (42) is invariant under translation of the independent variable and $w$ and $w'$ form a basis of first-order invariant). We obtain

$$y = 0 \text{ or } (c^2f(w) - g(w))\frac{dy}{dw} + (c^2f'(w) - g'(w)y + 2cf(w) = 0. \tag{43}$$

Equation (43b) is a linear equation that can be integrated using the standard integrating factor method to give

$$y = 2c(c^2f(w) - g(w))^{-1} \int^w f(r)(g(r) - c^2f(r))dr + \text{ const.} \tag{44}$$

3.2 $g(u) = cf(u) \exp\left[ m \int^u f(\theta)d\theta \right], \ m \neq 0,$

or $g(u) = cf(u) \left( \int^u f(\theta)d\theta + d \right)^m, \ m \neq 0, -4/3$

Here the nonzero Lie brackets of the two symmetry Lie algebras are $[Y_2, Y_3] = mY_2$ and $[Y_2, Y_3'] = mY_2$. These symmetry Lie algebras are realizations (modulo renaming and scaling the operators) of the Lie algebra $A_1 \oplus A_2$ in Patera and Winternitz [17] classification. The optimal systems of 1D subalgebras is $\Theta_1 = \{Y_1, Y_2, Y_3 + aY_1 (\text{ or } Y_3' + aY_1), Y_2 + \epsilon Y_1\}$, where $a \in \mathbb{R}$, and $\epsilon = \pm 1$. The only operators leading to invariant solutions not already treated are $Y_3 + aY_1$ and $Y_3' + aY_1$. It is more convenient to find invariant solutions of Eq. (7) and convert them into invariant solutions of Eq. (4) via the transformation $(6)$. We shall adopt this approach in the remainder of this section.
Solutions of Eq. (7) invariant under $X_3 + aX_1$ are given by

$$v = \frac{2}{m} \ln x + F(\tau x^{-a/m}) ,$$

where $F$ satisfies the ODE

$$\lambda^2(a^2c - 4m^2e^{-mF})F'' + c [a^2m\lambda^2F'^2 + a(a - 3m)\lambda F' + 2m] = 0,$$

where $\lambda = \tau x^{-a/m}$ and the prime stands for differentiation with respect to $\lambda$.

Similarity solutions of Eq. (7) stemming from the symmetry $X_3' + aX_1$ are given by

$$v = x^{2/m} F(\tau x^{-a/m}),$$

where $G$ satisfies the ODE

$$\lambda^2G(4m^2 - a^2c^2G'' - c[a^2m\lambda^2G'^2 + a(a - 3m - 4)\lambda G^{m+1}G' + 2(m + 2)G^{m+2}]) = 0,$$

and the prime stands for differentiation with respect to $\lambda$.

### 3.3 $g(u) = cf(u) \left( \int_u^\infty f(\theta) d\theta + d \right)^{-4/3}$

The nonzero brackets of symmetry operators are $[Y_2, Y_3'] = -(4/3)Y_2$, $[Y_2, Y_4] = (-1/4)Y_3'$, and $[Y_3', Y_4] = (-4/3)Y_4$. By letting $e_1 = Y_2$, $e_2 = (-3/4)Y_3'$, $e_3 = -3Y_4$, and $e_4 = Y_1$, we obtain a basis of the Lie algebra $A_{3,8} \oplus A_1$ in Patera and Winternitz [17] classification. An optimal system of 1D subalgebras of $A_{3,8} \oplus A_1$ is $\mathcal{O}_1 = \{e_1, e_4, e_2 + be_4, e_1 + e_3 + be_4, e_1 + ee_4\}$, where $a \geq 0$, $b \in \mathbb{R}$, and $\epsilon = \pm 1$. The only operator that will yield an invariant solution not yet considered is $e_1 + e_3 + be_4$ i.e. $Y_2 - 3Y_4 + bY_1$ or equivalently $X_2 - 3X_4 + bX_1$.

A solution of Eq. (7) invariant under $X_2 - 3X_4 + bX_1$ is given by

$$v = (3x^2 - 1)^{-3/2} H(\theta)$$

where

$$\theta = \tau \left( \frac{1 + \sqrt{3}x}{1 - \sqrt{3}x} \right)^k, \quad k = -b/(2\sqrt{3}),$$

and $H$ satisfies the ODE

$$\theta^2 H(H^{4/3} - 12k^2)H'' + 16k^2\theta^2 H'^2 - 12k^2\theta HH' - 9H^2 = 0,$$

where the prime stands for differentiation with respect to $\theta$. 
4 Separable solutions via group foliation

The group foliation method [15, 21] for obtaining particular solutions of partial differential equations (PDEs) consist in appending to a given PDE invariant differential constraints and solving the resulting overdetermined system (known also as the resolving system). Solutions of this system are obviously solutions of the initial equations, however the may not coincide with its invariant solutions. For this reason, these particular solutions are sometimes called partially invariant solutions. In this section, we follow Hu-Qu [9] group foliation based method for the construction of partially invariant separable solutions of Eq. (4). We start by supplementing equation Eq. (4) with the following first-order differential constraints invariant under $X_1 = \partial/\partial t$:

$$u_t = F(x, u), \quad u_x = G(x, u). \tag{52}$$

In order to obtain separable solutions of Eq. (4), after Hu and Qu [9], we let $G(x, u) = p(x)H(u)$. The integrability condition of Eq. (52) reads $F_x + u_u F_u = G_u u_t$ i.e. $F_x + p(x)H(u)F_u = p(x)H_u F$. The solution of the latter linear PDE is $F(x, u) = H(u)q(\alpha)$, where $\alpha = \alpha(t)$ and $\alpha_t = q(\alpha)$. Thus separable solutions (excluding the trivial case $H = 0$) are given by

$$\int^u_1 \frac{1}{H(r)} dr - \int^x p(y)dy = \alpha(t). \tag{53}$$

Thus in order to completely determine separable solutions, we have to find $p(x)$, $q(\alpha)$ and $H(u)$. The determination of these functions will force restrictions on $f(u)$ and $g(u)$. Substituting Eq. (52) into Eq. (4) and using the appropriate Ansätze for $F$ and $G$ yield

$$\left(H_u + \frac{f_u}{f} H\right) q^2 - \frac{g}{f} p_x = \left(\frac{g}{f} H_u + \frac{g_u}{f} H\right) p^2 + q_\alpha q + 2q = 0. \tag{54}$$

Note the change of variable $\bar{u} = \eta(u)$ is an equivalence transformation of the resolving system. Thus without loss of generality we may assume $H(u) = 1$ and Eq. (54) reduces to

$$\frac{f_u}{f} q^2 - \frac{g}{f} p_x - \left(\frac{g_u}{f}\right) p^2 + q_\alpha q + 2q = 0. \tag{55}$$

Differentiating Eq. (55) totally with respect to $x$ leads to

$$\left(\frac{f_u}{f}\right)_u p q^2 = \left(\frac{g}{f}\right)_u p_{xx} + \left[\left(\frac{g}{f}\right)_u + \frac{2g_u}{f}\right] pp_x + \left(\frac{g_u}{f}\right)_u p^3. \tag{56}$$

We consider the cases $(f_u/f)_u = 0$ and $(f_u/f)_u \neq 0$. 

Case 1: \( f = c_1 e^{a_1 u} \) where \( c_1 \neq 0 \) and \( a_1 \) are constants. Let \( \Gamma \) be the vector space generated by \( p_{xx}, pp_x \) and \( p^3 \). Assuming that \( p \neq 0 \), the dimension of \( \Gamma \) is either 1 or 2.

**Subcase 1.1:** \( \dim \Gamma = 1 \). In this case, \( p = p_0 \) or \( p = (p_1 x + p_0)^{-1} \), where \( p_0 \) and \( p_1 \neq 0 \) are constants.

In the first possibility viz. \( p = p_0 = \text{const.} \neq 0 \), we find after some calculations that

\[
\alpha = \begin{cases} 
\frac{1}{2} q + \frac{c_2 p_0^2}{4} \ln(2q - c_2 p_0^2) + \text{const.} & \text{if } a_1 = 0, \\
\frac{1}{2a_1} \ln |a_1 q^2 + 2q - c_2 p_0^2| - \frac{1}{a_1 \sqrt{-\Delta}} \arctan \left[ \frac{a_1 q + 1}{\sqrt{-\Delta}} \right] + \text{const.} & \text{if } a_1 \neq 0, \Delta < 0, \\
\frac{1}{2a_1} \ln |a_1 q^2 + 2q - c_2 p_0^2| - \frac{1}{a_1 \sqrt{\Delta}} \text{arctanh} \left[ \frac{a_1 q + 1}{\sqrt{\Delta}} \right] + \text{const.} & \text{if } a_1 \neq 0, \Delta > 0, \\
\frac{1}{2a_1} \ln |q + \frac{1}{a_1}| + \frac{1}{a_1 q + 1} + \text{const.} & \text{if } a_1 \neq 0, \Delta = 0,
\end{cases}
\]

where \( c_1 \) to \( c_3 \) are constants and \( \Delta = 1 + a_1 c_2 p_0^2 \). The corresponding separable solutions read

\[
u = p_0 x - k_1 e^{-2 t} + \frac{c_2 p_0^2}{2} t + \text{const.} \quad \text{if } a_1 = 0,
\]

\[
u = p_0 x - \frac{1}{a_1} \ln \left| \tanh(\sqrt{-\Delta}(t + k_1)) \right| - \frac{t}{a_1} + \text{const.} \quad \text{if } a_1 \neq 0, \Delta < 0,
\]

\[
u = p_0 x - \frac{1}{a_1} \ln \left| \tan(\sqrt{\Delta}(t + k_1)) \right| - \frac{t}{a_1} + \text{const.} \quad \text{if } a_1 \neq 0, \Delta > 0,
\]

\[
u = p_0 x - \frac{t + k_1}{a_1^2} + \frac{1}{a_1} \ln |t + k_1| + \text{const.} \quad \text{if } a_1 \neq 0, \Delta = 0.
\]

If \( p = (p_1 x + p_0)^{-1} \), we are lead to \( a_1 = 2 p_1 \) and \( g(u) = c_2 e^{a_1 u} \). However this case corresponds to a linearizable equation.

**Subcase 1.2:** \( \dim \Gamma = 2 \). In this case, \( p_{xx} = \lambda_1 pp_x + \lambda_2 p^3 \), where \( \lambda_1 \) and \( \lambda_2 \) are constants. After some calculations that involve replacing \( p_{xx} \) by \( \lambda_1 pp_x + \lambda_2 p^3 \) in Eq. (56) and separating with respect to \( pp_x \) and \( p^3 \), we find that \( g \propto f \). But this corresponds to the case where Eq. (4) is linearizable.

**Case 2:** \( (f_u / f)_u \neq 0 \). Solve Eq. (56) for \( q^2 \) to obtain

\[
q^2 = \frac{B}{A} \frac{p_{xx}}{p} + \left[ 3 \frac{B_u}{A} + 2B + \frac{B_{uu}}{A} + \frac{(AB)_u}{A} \right] p_x + \left[ \frac{B_{uu}}{A} + \frac{(AB)_u}{A} \right] p^2,
\]

where

\[
A = \frac{f_u}{f}, \quad B = \frac{g}{f}.
\]

Differentiate Eq. (62) with respect to \( t \) to obtain
\[ 2q_{q_0} = \left( \frac{B}{A} \right)_u \frac{p_{xx}}{p} + \left[ 3 \left( \frac{B_u}{A} \right)_u + 2B_u + \right] p_x + \left[ \left( \frac{B_{uu}}{A} \right)_u + \left( \frac{(AB)_u}{A} \right)_u \right] p^2. \]  

(64)

Substituting Eqs. (63) and (64) into Eq. (55) yields

\[ 2q = \left[ B - 4B_u - 2AB - \frac{3}{2} \left( \frac{B_u}{A} \right)_u \right] p_x - \left[ B + \frac{1}{2} \left( \frac{B}{A} \right)_u \right] \frac{p_{xx}}{p} + \left[ B_u + AB - B_{uu} - (AB)_u - \frac{1}{2} \left( \frac{B_{uu}}{A} \right)_u - \frac{1}{2} \left( \frac{(AB)_u}{A} \right)_u \right] p^2 \]  

(65)

Denote the expressions within the brackets in Eq. (65) by \( \zeta_1, \zeta_2 \) and \( \zeta_3 \). Differentiate Eq. (65) with respect to \( x \) to obtain

\[ (\zeta_{1,u} + 2\zeta_3)pp_x + (\zeta_1 + \zeta_{2,u})p_{xx} + \zeta_2 \left( \frac{p_{xx}}{p} \right)_x + \zeta_{3,u}p^3 = 0. \]  

(66)

Let \( \Lambda \) be the vector space spanned by \( pp_x, p_{xx}, (p_{xx}/p)_x \) and \( p^3 \). The constraints on \( f \) and \( g \) depend on the dimension of \( \Lambda \). Due to Eq. (66), \( \Lambda \) can be at most 3-dimensional. If \( \dim \Lambda = 1 \), then \( p \) is a nonzero constant, \( p_0 \), say, and \( \zeta_3 = \lambda_0 = \text{const.} \) In this case we obtain the separable solution \( u = p_0x + \lambda_0p_0^2t/2 + \text{const.} \)

It can be easily shown that \( \dim \Lambda = 2 \) if and only if

\[ p_{xx} = \lambda_1 pp_x + \lambda_2 p^3, \]  

(67)

where \( \lambda_1 \) and \( \lambda_2 \) are constants. A simple computation gives

\[ \left( \frac{p_{xx}}{p} \right)_x = (\lambda_1^2 + 2\lambda_2)pp_x + \lambda_1\lambda_2 p^3. \]  

(68)

Substitute Eqs. (67) and (68) into Eq. (66) and separate the latter with respect to \( pp_x \) and \( p^3 \) to obtain the following constraints on \( f \) and \( g \)

\[ \zeta_{1,u} + 2\zeta_3 + \lambda_1(\zeta_1 + \zeta_{2,u}) + (\lambda_1^2 + 2\lambda_2)\zeta_2 = 0, \]  

(69)

\[ \lambda_2(\zeta_1 + \zeta_{2,u}) + \lambda_1\lambda_2\zeta_2 + \zeta_{3,u} = 0. \]  

(70)

We could not solve Eqs. (69) and (70) to get \( f \) and \( g \). However Eq. (67) is exactly solvable since it admits the two-dimensional symmetry Lie algebra spanned by \( \partial_x \) and \( x\partial_x - p\partial_p \).

If \( \dim \Lambda = 3 \), then

\[ \left( \frac{p_{xx}}{p} \right)_x = a_1p_{xx} + a_2pp_x + a_3p^3, \]  

(71)
where \(a_1, a_2\) and \(a_3\) are constants. The functions \(f\) and \(g\) are constrained by

\[
\begin{align*}
\zeta_1 + \zeta_{2,u} + a_1 \zeta_2 &= 0, \quad (72) \\
\zeta_{1,u} + 2\zeta_3 + a_2 \zeta_2 &= 0 \quad (73) \\
\zeta_{3,u} + a_3 \zeta_2 &= 0. \quad (74)
\end{align*}
\]

Differentiate Eq. (72) with respect to \(u\), solve the resulting equation with respect to \(\zeta_{1,u}\), and substitute the solution into Eq. (73). Solve the resulting equation for \(\zeta_3\) and substitute the solution into Eq. (74) to obtain

\[
\zeta_{2,uuu} - a_1 \zeta_{2,uu} + a_2 \zeta_{2,u} - 2a_3 \zeta_2 = 0. \quad (75)
\]

The general solution of Eq. (75) depends on the roots of the cubic equation

\[
r^3 - a_1 r^2 + a_2 r - 2a_3 = 0. \quad (76)
\]

After determining \(\zeta_2\), \(\zeta_1\) and \(\zeta_3\) are easily obtained from Eqs. (72) and (73) respectively. We could not determine \(f\) and \(g\) from the expressions of \(\zeta_1\), \(\zeta_2\) and \(\zeta_3\).

5 Conclusion and discussions

We have completely classify the nonlinear hyperbolic heat equation from a symmetry standpoint. We found that it either has a two-, three-, four or an infinite-dimensional symmetry Lie algebra. In this last case, we showed that the hyperbolic heat equation is linearizable. We obtained invariant solutions and similarity reductions of the hyperbolic heat equation. In particular, we found steady state solutions i.e. solutions invariant under time translation and we characterized completely traveling wave solutions. Moreover, we found additive separable solutions using group foliation method.

References


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