Inextensible Flows of Curves in Minkowskian Space

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Abstract

In this paper we investigate inextensible flow of curves in Minkowski 3-space. Necessary and sufficient conditions for an inextensible curve flow are expressed as a partial differential equation involving the curvature and torsion.

Mathematics Subject Classification: 53C44; 53A04; 53A05

Keywords: Minkowski plane curve, Heat flow, Curvature flow, Inextensible

1 Introduction

It is well known that many nonlinear phenomena in physics, chemistry and biology are described by dynamics of shapes, such as curves and surfaces, and the evolution of curve and surface has significant applications in computer vision and image processing [15]. It has been known that a great number of nonlinear evolution equations are related to motion of curves in certain geometries [14, 13]. For instance, the Mullins’s nonlinear diffusion model of groove development [13] describes the curve shortening problem. Hasimoto [7] showed that the Schrödinger equation arises from motion of inextensible curves in $R^3$. 
The flow of a curve is said to be inextensible if the arclength is preserved. Physically, inextensible curve flows give rise to motions in which no strain energy is induced. The swinging motion of a cord of fixed length, for example can be described by inextensible curve flows. Such motions arise quite naturally in a wide range of physical applications. For example, both Chirikjian and Burdick [1] and Mochiyama et al. [12] study the shape control of hyper-redundant, or snake-like, robots. Inextensible curve flows also arise in the context of many problem in computer vision [8, 11] and computer animation [2], and even structural mechanics [16]. What the above problems share in common is the need to mathematically describe the inextensible time evolution of curves. There have been numerous studies in the literature on plane curve flows, particularly on evolving curves in the direction of their curvature vector field. Particularly relevant to this paper are the methods developed by Gage and Hamilton [4] and Grayson [6] for studying the shrinking of closed plane curves to a circle via the heat equation. In [5] Gage also studies area-preserving evolutions of plane curves. In [9, 10] Kwon et al. study inextensible flows of curves and developable surface in $\mathbb{R}^3$.

In this paper we investigate inextensible flow of curves in Minkowski 3-space. Necessary and sufficient conditions for an inextensible curve flow are expressed as a partial differential equation involving the curvature and torsion. We use some idea from [9, 10] in this paper.

## 2 Inextensible curve flows in Minkowski space

Let $V_3$ define a three-dimensional flat space with the line element

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu,$$

(1)

where $\mu, \nu = 1, 2, 3$, $x^\mu = (x, y, z)$ and $\eta_{\mu\nu} = \text{diag}(1, \epsilon, \epsilon)$. If $\epsilon = 1$, then $V_3 = E_3$ is a Euclidean 3-space and if $\epsilon = -1$ then $V_3 = M_3$ is a Minkowskian 3-space. Hence, Eq.(1) explicitly takes the form

$$ds^2 = dx^2 + \epsilon dy^2 + \epsilon dz^2.$$

Let $C$ be a curve on $V_3$ defined by $\alpha : I \rightarrow V_3$ and parameterized by its arc length $s \in I$. An orthonormal frame $(T^\mu, N^\mu, B^\mu)$ at each point of $C$ is defined by

$$\eta_{\mu\nu} T^\mu T^\nu = 1, \quad \eta_{\mu\nu} N^\mu N^\nu = \epsilon, \quad \eta_{\mu\nu} B^\mu B^\nu = \epsilon,$$
where \( T = \frac{d\alpha}{ds} \), all the other products vanish. The Serret-Frenet equations are

\[
\frac{dT^\mu}{ds} = kN^\mu, \\
\frac{dN^\mu}{ds} = -\epsilon kT^\mu - \tau B^\mu, \\
\frac{dB^\mu}{ds} = \tau N^\mu,
\]

where \( k \) and \( \tau \) are the curvature and the torsion scalar of the curve \( C \) at any point \( s \).

The vectors \( T, N, \) and \( B \) are, respectively, the tangent, normal and bi-normal vectors to the curve at any point \( s \) \([3]\). If \( \epsilon = -1 \) then we write \( \langle v, w \rangle \) for the value \( \eta(v, w) \).

A curve \( C \), locally parameterized by \( \alpha : I \subset \mathbb{R} \longrightarrow M_3 \), is said to be timelike, spacelike, or null curve if \( \langle \frac{d\alpha}{dt}, \frac{d\alpha}{dt} \rangle \) is positive, negative, or zero, respectively.

Throughout this article, we assume that \( F : [0, l] \times [0, \omega] \longrightarrow M_3 \) is a one-parameter family of smooth timelike curves in Minkowski space, where \( l \) is the arclength of the initial curve. Let \( u \) be the curve parametrization variable, \( 0 \leq u \leq l \).

The arclength of \( F \) is given by

\[
s(u) = \int_{0}^{u} | \frac{\partial F}{\partial u} | \ du
\]

where \( | \frac{\partial F}{\partial u} | = | \langle \frac{\partial F}{\partial u}, \frac{\partial F}{\partial u} \rangle |^{\frac{1}{2}} \). The operator \( \frac{\partial}{\partial s} \) is given in terms of \( u \) by

\[
\frac{\partial}{\partial s} = \frac{1}{v} \frac{\partial}{\partial u},
\]

where \( v = | \frac{\partial F}{\partial u} | \). The arclength parameter is \( ds = v du \). Any flow of \( F \) can be represented as

\[
\frac{\partial F}{\partial t} = fT + gN + hB.
\]

Letting the arclength variation be \( s(u, t) = \int_{0}^{u} v du \), in the Euclidean space the requirement that the curve not be subject to any elongation or compression can be expressed by the condition \( \frac{\partial}{\partial \tau} s(u, t) = \int_{0}^{u} \frac{\partial v}{\partial \tau} du = 0 \) for all \( u \in [0, l] \).

**Definition 2.1** A curve evolution \( F(u, t) \) and its flow \( \frac{\partial F}{\partial t} \) in \( M_3 \) are said to be inextensible if \( \frac{\partial}{\partial \tau} | \frac{\partial F}{\partial u} | = 0 \).

The necessary and sufficient conditions for inextensible flow in \( M_3 \) are then given by the following theorem.
Theorem 2.1 Let $\frac{\partial F}{\partial t} = fT + gN + hB$ be a smooth flow of the timelike curve $F$. The flow is inextensible if and only if $\frac{\partial f}{\partial s} = -gk$.

Proof: Since $F$ is timelike we have $v^2 = \langle \frac{\partial F}{\partial u}, \frac{\partial F}{\partial u} \rangle$. $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ commute since $u$ and $t$ are independent coordinates. So we have

$$2v \frac{\partial v}{\partial t} = \frac{\partial}{\partial t} \left( \left\langle \frac{\partial F}{\partial u}, \frac{\partial F}{\partial u} \right\rangle \right)$$

$$= 2 \left\langle \frac{\partial F}{\partial u}, \frac{\partial}{\partial u} \left( \frac{\partial F}{\partial t} \right) \right\rangle$$

$$= 2 \left\langle \frac{\partial F}{\partial u}, \frac{\partial}{\partial u} \left( fT + gN + hB \right) \right\rangle$$

$$= 2v \left( \frac{\partial f}{\partial u} T + f v k N + \frac{\partial g}{\partial u} N + g(kvT - vT B) + \frac{\partial h}{\partial u} B + hvN \right)$$

$$= 2v \left( \frac{\partial f}{\partial u} + gvk \right)$$

Thus we get

$$\frac{\partial v}{\partial t} = \frac{\partial f}{\partial u} + gvk \quad (2)$$

Now let $\frac{\partial F}{\partial t}$ be extensible. From (2) we have

$$\frac{\partial}{\partial t} s(u, t) = \int_0^u \frac{\partial v}{\partial t} du$$

$$= \int_0^u \left( \frac{\partial f}{\partial u} + gvk \right) du$$

$$= 0$$

for all $u \in [0, l]$. This implies that $\frac{\partial f}{\partial u} = -gvk$, or $\frac{\partial f}{\partial s} = -gk$. The argument can be reversed to show sufficiency, completing the proof. $\square$

We now restrict ourselves to arclength parameterized curves. That is, $v = 1$, and the local coordinate $u$ corresponds to the curve arclength $s$. We require the following lemma.

Lemma 2.1

$$\frac{\partial T}{\partial t} = \left( f k + \frac{\partial g}{\partial s} + h \tau \right) N + \left( -g \tau + \frac{\partial h}{\partial s} \right) B,$$

$$\frac{\partial N}{\partial t} = \left( f k + \frac{\partial g}{\partial s} + h \tau \right) T + \psi B,$$

$$\frac{\partial B}{\partial t} = \left( \frac{\partial h}{\partial s} - g \tau \right) T - \psi N,$$

where $\psi = \left\langle \frac{\partial N}{\partial t}, B \right\rangle$. 
Proof: Using the Frenet-Serret equations and Theorem 2.1, we calculate

\[
\frac{\partial T}{\partial t} = \frac{\partial}{\partial t} \frac{\partial F}{\partial s} = \frac{\partial}{\partial s} (fT + gN + hB) = \frac{\partial f}{\partial s} T + fkN + \frac{\partial g}{\partial s} N + g(kT - \tau B) + \frac{\partial h}{\partial s} B + h\tau N = \left( fk + \frac{\partial g}{\partial s} + h\tau \right) N + \left( -g\tau + \frac{\partial h}{\partial s} \right) B.
\]

Now differentiate the Frenet frame by \( \psi N \):

\[
\begin{align*}
0 &= \frac{\partial}{\partial t} \langle T, N \rangle = \left\langle \frac{\partial T}{\partial t}, N \right\rangle + \left\langle T, \frac{\partial N}{\partial t} \right\rangle = - \left( fk + \frac{\partial g}{\partial s} + h\tau \right) + \left\langle T, \frac{\partial N}{\partial t} \right\rangle, \\
0 &= \frac{\partial}{\partial t} \langle T, B \rangle = \left\langle \frac{\partial T}{\partial t}, B \right\rangle + \left\langle T, \frac{\partial B}{\partial t} \right\rangle = g\tau - \frac{\partial h}{\partial s} + \left\langle T, \frac{\partial B}{\partial t} \right\rangle, \\
0 &= \frac{\partial}{\partial t} \langle N, B \rangle = \left\langle \frac{\partial N}{\partial t}, B \right\rangle + \left\langle N, \frac{\partial B}{\partial t} \right\rangle = \psi + \left\langle N, \frac{\partial B}{\partial t} \right\rangle,
\end{align*}
\]

From the above we obtain \( \frac{\partial N}{\partial t} = (fk + \frac{\partial g}{\partial s} + h\tau) T + \psi B \) and \( \frac{\partial B}{\partial t} = (\frac{\partial h}{\partial s} - g\tau) T - \psi N \), since \( \langle \frac{\partial N}{\partial t}, N \rangle = \langle \frac{\partial B}{\partial t}, B \rangle = 0 \).

The following theorem states the conditions on the curvature and torsion for the curve flow \( F(s, t) \) to be inextensible.

**Theorem 2.2** Suppose the curve flow \( \frac{\partial F}{\partial t} = fT + gN + hB \) is inextensible. Then the following system of partial differential equations holds:

\[
\begin{align*}
\frac{\partial k}{\partial t} &= \frac{\partial}{\partial s} (fk) + \frac{\partial^2 g}{\partial s^2} + \frac{\partial}{\partial s} (h\tau) - g\tau^2 + \frac{\partial h}{\partial s} \\
\frac{\partial \tau}{\partial t} &= k \left( \frac{\partial h}{\partial s} - g\tau \right) - \frac{\partial \psi}{\partial s} \\
k\psi &= -\tau (fk + \frac{\partial g}{\partial s} + h\tau) - \frac{\partial}{\partial s} (g\tau) + \frac{\partial^2 h}{\partial s^2}
\end{align*}
\]

Proof: Noting that \( \frac{\partial}{\partial s} \frac{\partial T}{\partial t} = \frac{\partial}{\partial t} \frac{\partial T}{\partial s} \),

\[
\frac{\partial}{\partial s} \frac{\partial T}{\partial t} = \frac{\partial}{\partial s} \left[ (fk + \frac{\partial g}{\partial s} + h\tau) N + (-g\tau + \frac{\partial h}{\partial s}) B \right] = \left( \frac{\partial}{\partial s} (fk) + \frac{\partial^2 g}{\partial s^2} + \frac{\partial}{\partial s} (h\tau) \right) N + \left( fk + \frac{\partial g}{\partial s} + h\tau \right) (kT - \tau B) + \left( -\frac{\partial}{\partial s} (g\tau) + \frac{\partial^2 h}{\partial s^2} \right) B + \left( -g\tau + \frac{\partial h}{\partial s} \right) \tau N,
\]
while

\[
\frac{\partial}{\partial t} \frac{\partial T}{\partial s} = \frac{\partial}{\partial t} (kN) = \frac{\partial k}{\partial t} N + k \left[ (fk + \frac{\partial g}{\partial s} + h\tau)T + \psi B \right]
\]

Hence we see that

\[
\frac{\partial k}{\partial t} = \frac{\partial}{\partial s} (fk) + \frac{\partial^2 g}{\partial s^2} + \frac{\partial}{\partial s} (h\tau) - g\tau^2 + \tau \frac{\partial h}{\partial s}
\]

and

\[
k\psi = -\tau \left( fk + \frac{\partial g}{\partial s} + h\tau \right) - \frac{\partial}{\partial s} (g\tau) + \frac{\partial^2 h}{\partial s^2}.
\]

Since \( \frac{\partial}{\partial s} \frac{\partial B}{\partial t} = \frac{\partial}{\partial t} \frac{\partial B}{\partial s} \), we have

\[
\frac{\partial}{\partial s} \frac{\partial B}{\partial t} = \frac{\partial}{\partial s} \left[ \left( \frac{\partial h}{\partial s} - g\tau \right)T - \psi N \right] = \left( \frac{\partial^2 h}{\partial s^2} - \frac{\partial}{\partial s} (g\tau) \right)T + \left( \frac{\partial h}{\partial s} - g\tau \right) kN - \frac{\partial \psi}{\partial s} N - \psi (kT - \tau B),
\]

while

\[
\frac{\partial}{\partial t} \frac{\partial B}{\partial s} = \frac{\partial}{\partial t} (\tau N) = \frac{\partial^\tau}{\partial t} N + \tau \left[ (fk + \frac{\partial g}{\partial s} + h\tau)T + \psi B \right].
\]

Thus

\[
\frac{\partial \tau}{\partial t} = k \left( \frac{\partial h}{\partial s} - g\tau \right) - \frac{\partial \psi}{\partial s}
\]

No other new formulas are obtained from the relation \( \frac{\partial}{\partial s} \frac{\partial N}{\partial t} = \frac{\partial}{\partial t} \frac{\partial N}{\partial s} \).

3 Inextensible flows of Minkowski plane curves

Let us investigate with more details the case of torsionless timelike inextensible flows in Minkowski space.

for the sake of simplicity, let us chose our coordinate system such that the
Inextensible flows of curves in Minkowskian space

Inextensible flows of curves $F(s, t)$ takes place in the $(x, y)$-plane. Then, the Serret-Frenet equations yields

\[
\frac{dT}{ds} = kN \\
\frac{dN}{ds} = kT \\
\frac{dB}{ds} = 0 \tag{3}
\]

Choosing $B$ as the usual unit vector $\mathbf{k}$ in the $z$-direction, it remains to solve the two-dimensional system of differential equation for $T$ and $N$. As it can easily be verified, the general solution of (3) is given by

\[
F(s, t) = \left( \int_0^s \cosh(\theta(s)) ds + a, \int_0^s \sinh(\theta(s)) ds + b, 0 \right)
\]

where the function $\theta(s)$ is given in terms of the curvature $k = k(s)$ by $\theta(s) = \int_0^s k(s) ds + \phi$, with $a, b$ and $\phi$ being arbitrary constants.

References


Received: February 18, 2008