Numerical Experiments in Relativistic Phase Generation through Time Reversal

G. N. Ord and E. Harley
Departments of Mathematics and Computer Science, Ryerson University, Toronto, Ontario, Canada

R. B. Mann and Andrew Lauritzen
Departments of Physics and Computer Science, University of Waterloo, Waterloo, Ontario

Zenon Harley and Qin Qin Lin
University of Toronto
Toronto, Ontario

Abstract

In quantum mechanics, particles are commonly represented as wave-packets. However recent work related to Feynman’s ‘Chessboard Model’ has shown that to a certain extent the picture may be reversed. Particle waves may be formed by carefully pairing together portions of a spacetime path that are traversed in opposite directions with respect to macroscopic time. This essentially builds a field of particles and antiparticles that in the continuum limit is best described by a complex amplitude. In this picture, the equations of quantum mechanics appear as the direct result of a simple classical stochastic process without the necessity of a formal analytic continuation. We illustrate the emergence of complex amplitudes from paired paths with a series of numerical experiments. The experiments are explicit constructions of spacetime densities from single paths and they allow us to probe macroscopically reversible propagation originating from stochastic processes.

Keywords: Chessboard Model, Dirac Equation, Special Relativity, Simulation
Table 1: The relation between Classical PDE’s based on stochastic models, and their ‘Quantum’ cousins. Formal analytic continuation transforms one to the other, but then the stochastic basis for the equations becomes formal. The microscopic basis for the classical equations were investigated by Kac[6]. The analog in the quantum domain is the Feynman chessboard model that provides a sum-over-paths formulation. Underlying the classical equations is a fundamental exponential decay governed by a collision rate \( a \). In the quantum domain the exponential decay becomes a phase factor where the collision rate \( a \) is replaced by \( i \) times the Compton frequency \( mc^2/h \). This paper is about how this phase factor may be generated by a simple stochastic process.

1 Introduction

Both the Schrödinger and Dirac equations are wave equations describing the propagation of wavefunctions in time. Whether or not wavefunctions directly represent elements of an external reality is still an open question. However their use as elements in a probability calculus for quantum particles is universally accepted. Typically we look for solutions of wave equations and make contact with a particle picture through the measurement postulates. The ‘particle’ in quantum mechanics is then a derived concept, a localized wave or wave-packet.

It is interesting to contrast this situation with classical contexts for close relatives of the quantum equations. In Table(1), three equations are categorized as ‘Classical’. They are respectively a two-component form of the telegraph equations due to Kac[6], a second order form of the same equation and the limiting case \( (c \to \infty) \) of the diffusion equation. Solutions of these equations are probability density functions (PDF’s) that equilibrate in time. All three equations may be understood as phenomenological descriptions of ensembles of classical particles. As such they are not fundamental, but are based on kinetic theory.

With suitable choices of real constants, the equations in the right hand column labeled ‘Quantum’ are the same equations as those on the left except for the appearance of the unit imaginary. The quantum equations are a formal analytic continuation away from familiar classical equations. In the table \( a \to im \) or \( D \to iD \) transforms the classical to the quantum. The appear-
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The relation between classical and quantum equations through analytic continuation is exemplified by the two related harbingers of the respective domains. The exponential decay $e^{-at}$, an ever-present feature of stochastic processes in classical mechanics, signals the decay of initial probability densities towards equilibrium. The quantum analog of this in the relativistic domain is a phase factor $e^{\pm imt}$ (Compton phase) that does not decay but introduces a fundamental phase related to the ‘Zitterbewegung’ of the Dirac Equation. The phase factor arises from the inclusion of the rest mass term in the energy for a free particle. In the rest frame of a particle the zero-momentum wavefunctions contain only this rest energy frequency. States with non-zero momentum are obtained from zero-momentum wavefunctions by a Lorentz transformation. The absence of the Compton phase in Schrödinger’s equation is due to the non-relativistic approximation that ignores the rest mass. However, the usual Feynman kernel for the non-relativistic path integral:

$$K(b,a) = f(t) \exp\left(\frac{im(x_b - x_a)^2}{2(t_b - t_a)}\right)$$

contains the first order expansion of the Compton phase $e^{\pm imt}$ in $(v^2/c^2)$ with $v = (x_b - x_a)/(t_b - t_a)$. Thus, in the non-relativistic approximation, the relevant momentum-dependent phase factors we need arise from the Lorentz transformations of the Compton phase. As a result, we shall concentrate on the Compton phase itself. As a preview of what this paper is about the reader might like to run Applet 12 that shows a stochastic generation of the Compton phase. The idea is that just as we could simulate the exponential decay $\exp(-at)$ by considering ensemble averages of random walks equilibrating in time, so we can also generate the Compton phase by considering more structured walks in spacetime.

Notice that the presence of the Compton phase means that the solutions of the quantum equations are not probability density functions (PDFs) but ‘wavefunctions’. In conventional quantum mechanics, wavefunctions do not arise from kinetic theory considerations of ensembles of particles. They are considered to be fundamental objects. By contrast, the central theme of this paper is to illustrate that the Compton phase can have a simple stochastic origin.

In our approach, the extra ingredient needed to lift classical stochastic processes from the generation of PDF’s to the generation of wavefunctions
without invoking a formal analytic continuation is the extra degree of freedom that allows particles to move in either direction in the ‘time’ variable. The availability of both time directions adds two new qualitative features. The first is that a single particle path can cover a whole ensemble of forward-in-time paths, so one can associate a whole ensemble of paths with a single particle. The second new feature that we gain by allowing both time directions is the ability to propagate subtraction. This feature is not present in classical models since PDF’s are the result of counting processes that only add paths. The more paths there are, the larger the value of the PDF. However, particles that are allowed to move in either time direction effectively add anti-particles to the system. Since particles and antiparticles can annihilate each other, the existence of more paths does not necessarily mean an increase in the resulting wavefunction, and interference effects and the existence of nodes are made possible.

Although not a kinetic theory basis for quantum mechanics, Feynman’s path-integral formulation comes closest to considering wavefunctions as resulting from particle paths. In its usual non-relativistic form, each path contributes a complex number as a statistical ‘weight’ to a wavefunction which is itself an average over a large ensemble of paths. Intriguing and useful as this formulation is, neither the association of a phase with a path, nor the necessity of associating a whole ensemble of ‘histories’ with a single particle, is currently understood to be a consequence of an underlying dynamical process. Feynman paths are not ‘real’ in the sense that Brownian paths are thought to be ‘real’.

There is, however, one case where Feynman’s path integral approach hints at a more fundamental origin for the phase of a path. This is Feynman’s chessboard model[2, 3, 4, 7, 9], his path-integral approach to a relativistically correct description of a particle in a two dimensional spacetime. In this model, phase is fundamentally discrete and the continuum phase that carries over to the non-relativistic path-integral is the result of a statistical averaging of this discrete phase. Recent work on extending this approach by making use of space-time geometry to provide quantum interference effects shows that such models can be self-quantizing. If paths are paired with reversed spacetime orientation in such a way that they entwine (that is, symmetrically criss-cross) and form a chain of spacetime areas that alternate orientation, (see for example the sketch on the left in Figure (1) ) ensembles of such pairs can give rise to wave phenomena. In this paper we shall be experimenting (numerically) with the generation of various kinds of entwined paths. The objective is twofold.

At a general level, wave propagation in quantum mechanics is so ubiquitous that almost all treatments assume waves as a starting point. The ‘particle’ then becomes a wave-packet and only a wave-packet! This is despite the fact

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1The identification of ensembles of paths with single particles is a characteristic of the path-integral formulation of quantum mechanics. [2]
that in classical physics, on the scale of our senses, waves are usually just collective motions of particles. The experiments we describe in this paper are unusual and instructive in that they show graphically that it is possible to take the classical particle paradigm with piecewise smooth paths, allow time reversed segments, and end up with a form of wave propagation that requires none of the usual artifices of formal analytic continuation such as imaginary time, Wick rotation or canonical quantization. The experiments show that we can, both in principal and in practice, turn the tables and view waves as an approximation to the behaviour of a single particle allowed the freedom of both time directions.

At a more technical level, the original discovery that the Chessboard model had ontological roots in a single-path model [11] did not encourage a numerical exploration of the new paradigm. In part this was because the generation of propagators from chessboard paths required numerical operation counts that were of exponential order in time. Even with super-computer resources, the Dirac propagator could be simulated only out to a few Compton wavelengths[12]. Furthermore, chessboard paths are themselves embedded in a ‘sea’ of diffusive paths that make convergence of the propagator slow. However, the more recent discovery that one does not need a stochastic element throughout the entire spacetime region affected by a propagator[13] changes this picture quite drastically. Simulations in this paper are typically linear in time and convergence issues are considerably less of a problem. We shall also see that there is an interesting and fairly transparent relation between Poisson processes and the Compton phase.

2 Entwined Paths and Complex Numbers

Returning to the last row of Table 1, the Compton phase $e^{\pm imt}$ occupies a similar role in the solution of the quantum equations that the exponential decay $e^{-at}$ occupies in classical statistical mechanics. What we seek to illustrate is that by choosing particular geometries for the spacetime trajectory of a ‘stationary’ particle, we can see a simple origin for the Compton phase factor and the resulting implication of complex numbers in the solutions of our quantum PDEs. Thus in the rest of this article we shall be focusing on generating the Compton phase factor

$$e^{\pm imt}$$

via a variety of dynamical processes involving a single particle that is allowed to traverse a two dimensional spacetime many times. Here $m$ is some fixed positive constant that would represent a rest mass in the quantum context, but is simply a fixed inverse length in our simulations.

We work in a two dimensional space with Cartesian coordinates $(x, t)$. We
use the term ‘entwined path’ to mean any loop, or concatenation of loops, originating at the origin, in which each loop extends out past some maximum time $T$ and the forward and return portions of the loop continually cross each other. The idea is that the paired forward and return portions of the path form a chain of oriented spacetime areas in which the orientation switches at each intersection of the two paths. A sketch of this is found in Figure(1). The switching of orientation at each intersection of the two entwined paths may be associated with a single path that traverses the right-hand boundary of the oriented areas. We call this path an enumerative path because it allows us to count contributions of an entwined pair to a statistical averaging of orientation.

The enumerative path to the right of the entwined pair in Figure(1) has links displayed with two colours, blue and red, corresponding to direction of traversal in the associated entwined pair. It is the direction of traversal that defines the orientation of the associated spacetime area. Were we just counting paths such as the enumerative path, we would set up an indicator function defined on $(x, t)$, say $u_{\pm}(x, t)$ that would be either 1 or 0 depending on whether a path oriented in the $\pm x$ direction passed by the point $(x, t)$. We could imagine averaging this indicator function under suitable conditions on the allowed paths and in fact all the ‘classical’ equations in the left hand column of Table(1) may be obtained in such a fashion. However if we are going to allow our enumerative paths to count orientation, our indicator function has to have three possible values, 1, 0 or -1. Statistically averaging such a three-valued indicator function makes a big difference in terms of the type of equilibrium distributions that evolve in an ensemble average. In fact, all the equations in the right hand column of Table(1) can be obtained as ensemble averages of such a three-valued indicator function [14]. The last two graphs in Figure(1) sketch the indicator functions for the enumerative path to their left.

We can think of the three-valued indicator function as counting particles and antiparticles using Feynman’s picture of the relation between antiparticles and particles moving backwards in time[1, 17]. Thus the first density in the figure records the existence of a particle on the right light cone in the first time interval, no right-moving particle in the second time interval, a right moving antiparticle in the third interval and so on. The second density records the three-valued indicator function of left-moving particles. Note the normalization condition for this two-component density. The fact that the enumerative path is continuous and does not disappear leads to the condition that the sum of the squares of the two indicator functions is one. Were we only counting paths with a two-valued indicator function, the normalization would be simply that the sum of the indicator functions would be one. These different normalization conditions hint at the difference between the behaviour of the solutions of the the quantum and classical equations of Table(1). The classical equations yield solutions that are probability density functions; they are
non-negative and their integral over all $x$ is a constant. In comparison, the quantum equations have wave-like solutions that are square-integrable to a fixed value over $x$.

The qualitative features mentioned above of counting oriented areas using a three-valued indicator function with enumerative paths, suggests that we might be able to build the solutions of our quantum equations from sub-quantum dynamical processes provided that they are based on entwined paths that have some suitable geometry. This has been shown to be the case analytically [14] by relying on the known solutions of Feynman’s Chessboard model. However the picture of using entwined paths and oriented areas is more general than the specific case of the Chessboard model.

The numerical experiments reported below are designed to illustrate how entwining the forward and reversed paths allows one to reconstruct a physical analog of the Compton phase $e^{\pm imt}$ from the geometry of the path alone. A discussion of the mathematical models underlying the simulations are contained in the appendices. In Appendix A we discuss deterministic schemes for smoothing out the classical phase to produce approximations to $e^{\pm imt}$. In Appendix B we show how Poisson processes may be used to produce the same result.

The first numerical experiments show how to construct a deterministic process that will leave a ‘history’ of a three-valued density giving rise to an approximation of $e^{\pm imt}$, written on spacetime at $x = 0$.

Applet 1: Simulation of the entwined path and density of a single particle. The path and density are coloured blue when the traversal is parallel to $t$ and red when the traversal is antiparallel to $t$. The horizontal axis of the path is the x-coordinate of the entwined path (left). The horizontal axis on the right is a three-valued density. The vertical axis is time. View Applet 1
Applet 2: Simulation of the entwined paths and combined density of two paths, out of phase by a quarter-period. Two such paths are called a fiber. The paths have been chosen to build a square wave density illustrated on the right of the figure. The two paths are run in parallel rather than in series. View Applet 2

The experiments illustrate how the discrete phase of a particle allowed to move in both directions in the $t$-variable may be organized to mimic continuum phase. In the first experiment the method of assembly is completely artificial and designed to show that one can carefully choose an ensemble of entwined paths to build a good approximation to the Compton phase. The second experiment shows that a stochastic process can achieve the same result.

3 Experiment 1

Here we illustrate the relation between a simple deterministic entwined path and the counting of oriented areas that leads to the construction of phase. In Figure(1) a simple entwined path is sketched. The horizontal axis is a space axis and the vertical axis is ‘observer’ time. The particle has its own ‘time’ parameter which is not a single valued function of observer time. In the figure, portions of the trajectory in which both observer and particle time are increasing is coloured blue; when observer time is a decreasing function of particle time the path is coloured red. Note that the rectangles bounded by the entwined path have alternating orientation in that their boundaries are traversed in opposite directions. The geometry of the entwined path is motivated by special relativity – we assume that all particles have the same characteristic speed $c$, where we shall set $c = 1$ for convenience. Choice of this characteristic speed is motivated by the uncertainty principle which forces high momentum and hence high velocity on small scales. Since $c$ is the maximum physical speed allowed we construct paths with this speed on the smallest
scales. The frequency of the entwinement is fixed by the ‘rest mass’ of the particle, and all rectangles in the chain of rectangles formed are identical. Returning to Figure(1) the slopes of the line segments are all ±1 and all steps are light-like. In counting the oriented areas in the figure, the first rectangle may be recorded as a plus one, assuming a counterclockwise traversal is positive. The second rectangle then contributes a minus one, and so on. Notice that we can get the sign of the contribution by looking at the path formed to the right of the crossing points of the entwined path. Although this is not the trajectory taken by the particle, we call the path an ‘enumerative path’ because it is this path that helps us enumerate the oriented areas. In one dimension the enumerative path is made of links that are either parallel to the (1,1) direction or perpendicular to it. We thus count the oriented contribution of a path in two components, a (1,1) component and a (-1,1) component.

The two densities in the figure then count the orientations of the chain of areas formed by the path, partitioned across the two directions. Notice that after a single cycle of the particle from the origin out to the maximum time and then back again, the resulting densities are interleaving square waves. This is simulated, keeping track of the (1,1) density in Applet (1). To build an approximation to the Compton phase we shall concatenate entwined paths that have had their origin shifted in the +t direction, first to build square waves, then to build alternating delta functions, and finally to build the trigonometric functions. The process of extracting square waves and delta functions from paths is illustrated in Figures 1 and 2. Animations of this are contained in Applets(2,3,4). Animations of the construction of approximations to the trigonometric functions appear in Applets(5,6).

Applet 3: Simulation of the entwined paths and combined density of two fibers. One fiber is shifted by a half-period plus epsilon relative to the other fiber. This arrangement of two fibers (equivalent to two particles and two antiparticles) is called a cord. Notice the densities cancel each other out except at a periodic succession of spikes of alternating sign. View Applet 3
Note that the phase shift of $\pi/2$ between the two components of $e^{int}$ is guaranteed by the geometry of the entwined path and we have omitted the imaginary component from the figures. The analytic description of the density evolution is relegated to Appendix A.

Applet 4: Simulation of a cord (as in Applet 3) except that forward and reverse densities are added simultaneously so the cancellation is immediate and the constructive interference shows up as periodic spikes. The alternating periodic delta functions may then be scaled, shifted and added simply by shifting the entwined path giving rise to them. View Applet 4

Applet 5: A discrete approximation to a sine wave is shaped from many cords (see text). In the figure, $N=10$, meaning that there are 10 positions for delta functions in a half period. At each position there are enough delta functions to approximate the shape of the sine wave. View Applet 5

Applet 6: The number of cords is increased to increase the resolution of the resulting sine wave. The actual paths are omitted as they would be too dense to resolve on screen. View Applet 6
As the simulation progresses from Applet(1) through Applet(5) the source points of the entwined paths are varied by hand. That is, they are chosen deliberately to fashion a discrete approximation to the required trigonometric functions. We are not trying to mimic a natural process of construction here; instead we are verifying that concatenation of entwined paths and the resulting oriented areas are sufficient to approximate the trigonometric functions necessary to describe the Compton phase. In the second set of experiments, we allow a stochastic process to vary the source points. This provides a more natural origin for the formation of the trigonometric functions.

4 Experiment II

In the previous experiment, the discrete approximation to the complex exponential was formed in a transparent but artificial way. We simply built the trigonometric functions via alternating delta functions in a deterministic procedure. The objective was to show that the particular geometry of entwined paths was sufficient to produce a discrete approximation to the Compton phase factor, illustrating the role of projections onto the two light cones and the two dimensions of the complex number.

A less artificial process would be to have the first few cycles be stochastically chosen in such a way that the ‘classical’ exponential decay factor and the Compton phase are driven by a common stochastic process. By ‘first few cycles’ here we mean the first rectangular spacetime areas formed by the path as it (repeatedly) originates from a region about $t = 0$. Figure 4 illustrates the idea.

There are many ways to choose a stochastic first cycle. One method is to choose a stochastic phase for the particle at each traversal of the region near the origin. By stochastic phase here we mean that we introduce a stochastic element to where the first deterministic oriented area starts. In this experiment, discrete random walks that approximate the Poisson distribution for events per unit time are chosen with a time constant that matches the wavelength of $e^{imt}$. In Applet(7) a stochastic first cycle is used to ‘start’ a deterministic entwined path. In Applets(8) (9) and (10) we see the accompanying enumerative path densities to one, two and thirty enumerative paths respectively. Note that when running the applets, pushing the reset button redraws the simulation. To get a new random first cycle, close the applet window and start again. Applets 8 through 10 include the transient contribution of the first cycle that contributes to an exponential decay.
Applet 7: An entwined path with a stochastic first cycle. The embedded picture and that generated by the applet will differ as the applets generate different stochastic paths. View Applet 7

Applet 8: An entwined path with a stochastic first cycle and its right-moving density. Forward/reverse paths are run in parallel. View Applet 8

Applet 9: Two entwined paths with stochastic first cycles and the resulting density. View Applet 9

Applet 10: Thirty entwined paths with stochastic first cycles and the resulting density. The density is noisy when only a few paths are involved. The next applet shows the emergence of the underlying signal when more paths are involved. View Applet 10
Applet (11) shows a simulation of the entwined path generation process solved exactly in Appendix B. The solid lines in the first two graphs in the figure correspond to the expected values from the appendix. The points accumulating in the simulation are the results extracted from the stochastic process as the entwined paths repeatedly traverse the spacetime region. The third graph in the simulation sums the squares of the two component wavefunctions in each elementary time interval.

Applet 11: This is a simulation of the continuous time stochastic process analyzed in appendix B, with the $t$-variable on the horizontal axis. The exponential decay in the first cycle has been filtered out and the amplitude normalized to 1. The upper and middle figures are the projections onto the left and right light cones respectively. The bottom graph shows the sum of the squares of the upper two densities. These are one for a single path, stochastic for many paths but approach 1 as the number of runs increases.
Applet 12: This is a simulation of the continuous time stochastic process analyzed in appendix B, with the components of the ‘wavefunction’ plotted as a point in a two-dimensional space. The vertical axis is $t$, the two horizontal axes register the left and right light cone densities. The results from both the right and left enumerative path are plotted giving two helices that wind in opposite directions. The first frame shows the result of a single entwined pair. Subsequent frames show how repeated loops of entwined paths approach the ‘equilibrium’ spirals of the Compton phase in Table 1. As in the previous simulation, the first frame shows a single path with the number of paths doubling with each frame.

The time interval for the displayed result is 1/400 of the total time interval. Deviations of the sum from 1 give an indication of the convergence of the wavefunctions to their expected value. The periodic nature of the fluctuations reflect the way the stochastic process assembles the signal. In particular, the local maxima in the error are at points where the Poisson process has to smooth the abrupt end of the interaction region. The error decreases as the number of runs increases; however, past a certain point the truncation error involved in resolving the signal into only 400 bins dominates and little is gained from further runs.
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Applet 13: This is the simulation of the process in Applet 12 viewed from above, i.e., looking down the $t$-axis. The first frame corresponds to a single entwined path. The line segments are linear interpolants used by the plotting routine, the actual data are the points where the line-segments join. The first frame shows only four occupied 'bins' corresponding to the four points at the corners of a square representing the four spacetime directions. The early evolution is noisy until all of the (400) bins of the numerical simulation are occupied. The late evolution makes sure all the bins are well-covered and the stochastic process approaches the expected circular spiral.

Applet 12 shows the evolution of the signal by using the left and right components as projections onto two orthogonal axes perpendicular to the $t$-axis. The blue curve is that obtained from the enumerative path consisting of the right-hand boundary of the entwined paths. The red curve is obtained from the left-hand boundaries. The first frame corresponds to a single entwined path and a comparison of this with the entwined path on the left in Figure (1) shows how the three-valued indicator functions map the entwined path onto a pair of discrete spirals in three dimensions. The discrete four-valued phase is clearly evident with a single entwined path but, as the Poisson process equilibrates the four intrinsic direction phases, the discrete spirals evolve into smooth helices that wind in opposite directions. The evolution of the smoothing process is particularly evident in Applet 13 where the view is from the top. As the Poisson process progresses, the two spirals are pulled out to form a circle.

5 Discussion

The distinction between the classical and quantum equations in Table(1) would be cosmetic were it not for the complex algebra induced by the presence of
$i$ in the quantum versions. The unit imaginary has the effect of completely changing the character of solutions from decaying exponentials to oscillatory wavefunctions. While conventional stochastic processes provide a sensible basis for the classical equations, there is no analog of ‘quantum interference’ in such processes. As a result, attempts to use classical stochastic processes as a basis for quantum processes always have to invoke phase as a ‘background feature’ necessary to mimic wave propagation.

This work illustrates an alternative approach in which phase is not a background feature but a manifestation of microscopic paths that are allowed time-reversed portions. In this picture, either a deterministic or stochastic process is used to ‘smooth-out’ the four intrinsic phases of the four possible spacetime directions, producing a dynamic construction of $e^{\pm imt}$. Unlike conventional approaches, the underlying dynamical process is ontological in the sense that the resulting waves are a real manifestation of the spacetime geometry of the underlying paths. The simulations in this paper illustrate this. In the first experiment we showed a deterministic construction of the components of $e^{\pm imt}$ to illustrate the fact that deterministic entwined paths with their discrete phase are sufficient to approximate the required phase factor to arbitrary precision. The second experiment showed that the signal can be assembled more naturally as the result of a Poisson process. In this case there was an interaction region near the origin where the direction changes were stochastic and driven by a Poisson process. The result was that the Poisson process assembled the relevant signal with no need for the detailed construction of the first experiment.

In contrast to the presence of phase in quantum mechanics, the Compton phase here is readily interpreted. In a classical diffusive system we would detect a particle’s path through spacetime by a Bernoulli random variable $X$ say where:

$$X = \begin{cases} 
1 & \exists \text{ particle in local spacetime volume} \\
0 & \text{otherwise.}
\end{cases}$$

(3)

However entwined paths oscillate about their bound partner and the random variable that measures the presence of the associated oriented area is three-valued. If we represent this as $Y$ we have:

$$Y = \begin{cases} 
1 & \exists \text{ a positive area in the local spacetime volume} \\
-1 & \exists \text{ negative area in the local spacetime volume} \\
0 & \text{otherwise}
\end{cases}$$

(4)

Whereas statistical averaging of $X$ can lead to probability density functions in the normal way, the same statistical averaging of $Y$ will yield density functions that are not positive-definite and cannot be regarded as probabilities. However
they are nonetheless density functions and judicious choice of the initial conditions and the geometry of the paths allow them to mimic the Compton phase. The convenient emergence of complex numbers results from the oscillation of path pairs rather than an invocation of the unit imaginary. (In particular see Equations (33) through (37))

The extraction of the Compton phase factor from the Poisson process and entwined paths illustrates the fact that some of the interpretive problems in quantum mechanics may be an artifact of the continuum language that we use. In conventional quantum mechanics one starts in the continuum where the Compton (or Feynman) phase is assumed continuous. Note that in the above formulation this would correspond to ignoring the underlying dynamical process and starting with the assumption that only the evolved density functions were important. This removes a layer of detail from the problem, but at the same time makes it non-recoverable from the mathematical formulation. Watching the evolution of the Compton phase in Applet 12 we can see how the simple 4-state stochastic process gives rise to the smooth spirals of the final frame. However, starting from those smooth spirals, which is where quantum mechanics starts, we would be hard-pressed to immediately guess that there is a simple dynamical process generating them. It is hoped that by simulating simple quantum systems using underlying dynamical processes, we may gain some insight into the relation between this particle oriented propagation and the measurement postulates.²

A The Deterministic Construction of Experiment 1

To motivate the deterministic construction we first consider Feynman’s Chessboard model [2, 3, 4, 5, 7, 10]. This was Feynman’s path-integral prescription for a propagator for the Dirac equation. Although the construction was completely stochastic, by examining it we can see how quantum interference is generated. In units with \(\hbar = c = 1\) Feynman’s kernel for propagation from a

² Readers familiar with the Feynman Chessboard model will notice that all the simulations in this paper rely on the fact that we are working in a two dimensional spacetime where a particle’s light-cone consists of only four spacetime directions. It is precisely this situation that allows Feynman’s Chessboard model to work, yet there is as yet no agreement as to the best way to extend the Chessboard model to 3+1 dimensions. However, there is strong evidence that the simplest dimensional extension of the Chessboard model is essentially an embedding that completely preserves the features that our stochastic models need[8, 16]. A demonstration of this is in preparation. It is also of interest to note that the non-relativistic free particle is relatively easily adapted to our simulations since in the non-relativistic approximation the three components of the momentum operator commute, making the extension from one dimension simpler.
to $b$ is:

$$K(b, a) = \sum_R N(R)(i\epsilon m)^R$$

$$= \left( \sum_{R=0,4,...} N(R)(\epsilon m)^R - \sum_{R=2,6,...} N(R)(\epsilon m)^R \right)$$

$$+ i \left( \sum_{R=1,5,...} N(R)(\epsilon m)^R - \sum_{R=3,7,...} N(R)(\epsilon m)^R \right)$$

$$= \Phi_+ + i \Phi_- .$$

where the sum is over all $R$-cornered paths with step size $\epsilon$ and $N(R)$ is the number of $R$-cornered paths. In Eq(5) we see that $i = \sqrt{-1}$ serves two purposes. It distinguishes right from left through the partition of the sums into real and imaginary components $\Phi_{\pm}$. It also constructs interference effects by making the sums over even and odd $R$ alternating. As has been pointed out in previous work, [11, 14, 15] the alternating subtractions can be given a realistic basis by covering the chessboard ensemble of paths by a single ‘Entwined Path’ (EP) that traverses the subtracted portions of the chessboard paths in the $-t$ direction, thereby giving the interference effects a simple origin in time-reversed portions of the path.

Following previous work [13] we facilitate removal of the stochastic component by constructing deterministic EP’s where the geometry of each closed loop is identical within a given path. We call such deterministic paths ‘velocity eigenpaths’. A $v = 0$ eigenpath is sketched with the associated right ‘enumerative path’ over a period of a single cycle in the left side of Figure 1. As in the stochastic case we need only consider the left or right ‘enumerative path’ for counting paths. Each enumerative path contributes alternately to right-moving and left-moving particles, the contribution being $\pm 1$ depending on the direction in $t$ of the original entwined path. If we look, for example, at the right-moving states, the single right envelope creates a square-wave density with periodic gaps where the particle has switched direction. Density is defined as the total number of paths through each spacetime point weighted by $+1$ for forward in time and $-1$ for backward in time. We count paths according to direction because each small loop can be interpreted as a creation of a virtual pair followed by an annihilation of the same pair at the end of the loop. We only use one envelope for counting because, in one-dimension, the other envelope contains the same information. We call right-moving particles on the enumerative path adolescent (just post-creation) and left moving particles senescent (pre-annihilation). The relationship between these two populations has an interesting feature due to the geometry of the paths. The adolescent density for the sketched path is:
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where we have not specified boundaries for $t$. In Figure(1), $t$ starts at $t_0 = 0$ and ends after one cycle at $t_R = 4$ but in the general non-relativistic case $t_R$ will be very much greater than the basic cycle time. Note that the senescent density for our path is simply $u_{S}(t) = u_{A}(t - 1)$.

To make contact with Feynman’s phase, we would like to make the densities drawn by our entwined path more obviously connected to $e^{\pm im t}$, since these are essentially the carrier waves generated in the Chessboard model. To do this deterministically we shall consider a three-stage process in the construction of a suitable entwined path. We first of all construct a ‘fibre’, a single entwined pair as illustrated in Figure(1). We then concatenate two fibres to produce a square wave density as in Figure(2). By concatenation here we mean literally joining the paths at or near the origin to produce a single path that may be traversed continuously from end to end. To concatenate paths that are out of phase with each other requires changing the size of the first oriented area, or simply shifting the origin along the $t$ axis. We shall not specify how this is done, we just note that it can be done in several ways, one of which is demonstrated in appendix B. We shall, however, use this degree of freedom to overlay paths in such a way as to smooth out the transitions between positive and negative orientation. Concatenating four paths as in Figure(3) produces a ‘cord’ that in turn creates an alternating periodic sequence of delta functions on spacetime, as in the left half of Figure(6). Finally we concatenate cords to build a ‘cable’ in such a way that the resulting spacetime densities approximate $e^{im t}$. At all steps in this process we add paths by concatenation . . . the resulting paths from fibre to cable form a single continuous spacetime path.

Proceeding, notice that if we concatenate two fibres shifted by one quarter period (Figure(2)) the resulting spacetime densities for the two right-envelopes are square waves. For the adolescent density we have:

\[
W_a(t) = u_a(t) + u_a(t - 1) \tag{7}
\]

The density of senescent particles is then just $W_a(t - 1)$. If we choose a lattice spacing $\epsilon = 4/(2n)$ we can convert the square-wave densities to a periodic array of lattice delta-functions by concatenating two more fibres shifted one lattice spacing away from a half-period shift of the original two. This will give an adolescent density function:

\[
\delta_a(t) = W_a(t) + W_a(t + 2 - \epsilon) \tag{8}
\]

This is pictured on the left in Figure(6). It is a periodic chain of discrete delta functions with alternating signs. The alternating signs allow us to simply
concatenate paths, while constructing a real density in which \textit{subtraction} plays an important role. Adding paths in the Wiener context does not allow for this, and cannot accommodate self-interference as a result. The same concatenation that produces $\delta_a(t)$ for adolescent particles automatically produces $\delta_s(t) = \delta_a(t - 1)$ for senescent particles.

Now we repeat the concatenation process shifting the temporal origin by $\epsilon$ each time. We do this $n$ times and at the $k$'th shift we concatenate $\|M \sin(\pi k/n)\|$ cords. Here $M$ is a large integer chosen to extract a specified accuracy from the trigonometric function. The resulting density is as sketched, for $n = 10$, $M = 20$ in Figure(6) on the right. The senescent density is identical in form but lags in phase by $\pi/2$.

It is clear from this procedure that we can get as close as we like to a representation of the trigonometric functions over any fixed number of cycles, simply by carefully choosing the EP concatenations. It should also be clear that there are many ways of assembling a cable to draw the two components of $e^{-imt}$ on spacetime. We have chosen a procedure that is transparent in its ultimate outcome, but unlikely to occur in the real world. However, the point here is not the artificiality of the construction method, it is the fact that it is a method that \textit{could} be used by a single point particle with no non-classical properties, outside of the extra degree of freedom in time, to \textit{encode} quantum information in spacetime using only a counting process.

To see how our scheme relates to the path-integral we recall that in the non-relativistic limit, \((x/t << 1, \ t >> 1/m)\) the un-normalized propagator from the Chessboard model may be written:

\[ K(x, t) = \exp(-imt(1 - x^2/2t^2)) \]  \hspace{1cm} (9)

up to a phase factor that is independent of $x$. The non-relativistic propagator is simply a low frequency signal sitting on top of the high-frequency zitterbewegung generated at the Compton frequency $m$. If we write this in terms of \(v = x/t\) this is

\[ K(x, t) = \exp(-imt) \exp(i \frac{mv^2}{2} t) \]  \hspace{1cm} (10)

and we see that along a ray of constant $v$ we have a simple complex exponential with a frequency reduced by a term proportional to the classical action along the path. We can thus construct, along this ray, an EP approximation to the complex phase factor simply by ‘writing’ along the ray with a suitable EP.
The limiting continuous-time stochastic process

The stochastic processes modeled in these numerical experiments are discrete-time, discrete-state processes. In all sample paths, space and time intervals are finite and there are just four possible states.

One simplification that illuminates the numerical work is to consider the continuous-time limiting process. In the continuous-time limit the underlying stochastic process in the interaction region is Poisson, and the emergence of the trigonometric functions is more direct. What happens is that if we write the partition function for the probability mass function (PMF) of the Poisson distribution we get:

$$Z = 1 = \exp(-\lambda t) \sum_{n=0}^{\infty} \frac{\lambda^n t^n}{n!}$$  \hspace{1cm} (11)

The enumerative paths partition the sum on the right into 4 classes corresponding to whether $n$ is 0, 1, 2, or 3 mod 4. i.e.

$$1 = \exp(-\lambda t) \left( \sum_{n=0 \atop n \equiv 0 \mod 4}^{\infty} \frac{(\lambda t)^n}{n!} + \sum_{n=1 \atop n \equiv 1 \mod 4}^{\infty} \frac{(\lambda t)^n}{n!} + \sum_{n=2 \atop n \equiv 2 \mod 4}^{\infty} \frac{(\lambda t)^n}{n!} + \sum_{n=3 \atop n \equiv 3 \mod 4}^{\infty} \frac{(\lambda t)^n}{n!} \right)$$  \hspace{1cm} (12)

However states 2 and 3 both contribute negative signs to expected orientation (or alternatively, 2 and 3 are anti-particle states) so the expected orientation in the interaction region is

$$E = \exp(-\lambda t) \left( \sum_{n=0 \atop n \equiv 0 \mod 4}^{\infty} \frac{(\lambda t)^n}{n!} + \sum_{n=1 \atop n \equiv 1 \mod 4}^{\infty} \frac{(\lambda t)^n}{n!} - \sum_{n=2 \atop n \equiv 2 \mod 4}^{\infty} \frac{(\lambda t)^n}{n!} - \sum_{n=3 \atop n \equiv 3 \mod 4}^{\infty} \frac{(\lambda t)^n}{n!} \right)$$

$$= \exp(-\lambda t) (\cos \lambda t + \sin \lambda t)$$  \hspace{1cm} (13)

Outside the interaction region the exponential decay in (13) ceases because states are no longer being mixed by a random process. The periodic form of the entwined paths outside the interaction region then propagates the trigonometric part of the signal.

The above sketch suggests what is happening physically. The two trigonometric functions in (13) are amplitudes that are projected onto the two orthogonal light-cones and the result is a two-component object that rotates with frequency $\lambda$. To show this mathematically we need to be a bit more precise. To this end we consider the experiment illustrated in Fig.(4). There is an interaction region corresponding to $0 < t \leq 1$. In this region events correspond to corners in the enumerative paths, Fig.(4 B). The distribution of waiting times between events is exponential and so the events themselves are Poisson distributed. Once a particle has escaped the interaction region
it executes the simple deterministic zig-zag trajectory of the figure, returning after some large predetermined value of $t$.

For simplicity we match the decay rate to the size of the interaction region choosing the rate constant $\lambda = \pi/2$. This means that the interaction region corresponds to one quarter of the resulting Compton wavelength. The sample paths have 4 states labeled $1 \ldots 4$ corresponding to $(x,t)$ traversal directions $(1,1), (-1,1), (-1,-1) & (1,-1)$ respectively as in Fig.4C).

Let $W_n(t)$ denote a sample path of state numbers for an entwined path after $n$ traversals of the relevant spacetime region. Since all such paths are loops that begin and end at the origin, the process is manifestly ergodic and counting contributions from $W_n(t)$ in the limit $n \to \infty$ is equivalent to ensemble averages of enumerative paths. As a result we can replace the dynamical process that consists of a single trajectory that covers the spacetime region many times, with an ensemble average of enumerative paths. Thus let $X(t)$ be a random process defined on an interval $[0, T]$. $X$ takes on values $X \in \{1, 2, 3, 4\}$ corresponding to the four ‘direction states’ of the enumerative path. In terms of counting contributions to the propagator, this corresponds to right-moving particle, left-moving particle, right-moving antiparticle and left-moving antiparticle respectively.

Our initial condition shall be that all paths start in state 1 at the origin so we define the PMF $P_X(k, t)$ at $(t = 0)$ to be:

$$P_X(k, 0) = \begin{cases} 
1 & k = 1, \\
0 & \text{otherwise}
\end{cases} \quad (14)$$

Here $k$ labels the states $1 \ldots 4$. To find the PMF at other times, we consider separately the interaction region $0 < t \leq 1$ where a stochastic process is operative, and later time intervals where each path is deterministic.

**Interaction region $0 < t \leq 1$**

In this region, walks are subjected to a Poisson process of rate $\lambda$. Each event changes the spacetime orientation of the path segment and changes the state of the enumerative path to the next in the sequence $1 \to 2 \to 3 \to 4 \to 1 \ldots$. Particles leave the region in whatever state they were in after the last event in the region. The only exception to this are the particles that traverse the entire region without any event. Such particles are reflected back to the origin and are recycled.

Given the PMF at $t = 0$ we can write down the PMF of $X(t)$ for any $t$ in the interaction region. For example, at time $t \leq 1$, the particle is in state 1 ($X(t) = 1$) iff there have been 0 (mod 4) events in the interval $(0, t]$. To write
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this compactly define the functions:

\[ Q_k(t) = \sum_{j=0}^{\infty} (\lambda t)^{k+4j} (k+4j)! \]  

(15)

We note the following identities:

\[ \sum_{k=0}^{3} Q_k(t) = e^{\lambda t} \]  

(16)

\[ Q_0(t) - Q_2(t) = \cos \lambda t \]  

(17)

\[ Q_1(t) - Q_3(t) = \sin \lambda t \]  

(18)

\[ Q_4(t) - Q_2(t) = \cos \lambda t - 1 \]  

(19)

Since we wish to keep track of those particles that escape the interaction region we shall count only those particles that have at least one event in the interaction region. Thus the PMF of particles in state 1 is then:

\[ P_X(1, t) = e^{-\lambda t} Q_4(t) \]  

(20)

This reflects the fact that in the interaction region a particle can be in state 1 after at least one event iff it has had 4, 8, \ldots events. Similarly, the remaining PMF’s are:

\[ P_X(k, t) = e^{-\lambda t} Q_{k-1}(t) \quad k = 1, 2, 3 \]  

(21)

Now states 1 and 3 correspond to spacetime traversals in opposite directions, as do states 2 and 4. It is the difference in occupation densities that we require. Define the ‘wavefunctions’

\[ \psi_R(t) = P_X(1, t) - P_X(3, t) \]  

\[ = e^{-\lambda t}(Q_4(t) - Q_2(t)) \]  

\[ = e^{-\lambda t}(\cos \lambda t - 1) \]

and

\[ \psi_L(t) = P_X(2, t) - P_X(4, t) \]  

\[ = e^{-\lambda t}(Q_1(t) - Q_3(t)) \]  

\[ = e^{-\lambda t} \sin \lambda t \]  

(22)

Notice that both components have an exponentially decaying amplitude superimposed on a sinusoidal term. \( \psi_R \) also contains a straight exponential decay. This corresponds to the paths that do not interact at all in the region and are ultimately recycled. Once a particle has escaped the interaction region
its path is deterministic and the PMF has to be calculated by reference back to
the interaction region. We now consider the region right after the interaction
region.

**First deterministic region** $1 < t \leq 2$

This region is outside the interaction region and all state changes are deter-
mministic. Each path changes state exactly once in the region and the PMF’s for
each state count contributions from two distinct populations. For example, if
a path contributed to state 1 in this interval, then it either entered the interval
in state 1 or it left the interval in state 1. It cannot do both because it has
exactly one state change in the interval. Denote the PMF of those particles
that are lifted into state 1 in the interval by $P^t_X(1, t)$ and those particles that
are lifted out of state 1 in the interval by $P^*_X(1, t)$. The calculation for each of
these terms is similar. For example, if a path contributes to $P^t_X(1, t_0)$ then it
entered the interval in state 4 at $t = 1$ and changed to state 1 in the interval
$(1, t_0)$. Since all state changes in $(1, t_0)$ are deterministic and based on the last
state change in $(0, 1)$, the path must have had its last stochastic change prior
to $t = t_0 - 1$. Thus $P^t_X(1, t_0)$ counts those paths in state 4 at time $t_0 - 1$ that
had no stochastic events in the interval $(t_0 - 1, 1]$. Thus:

$$P^t_X(1, t_0) = P_X(4, t_0 - 1)e^{-\lambda(2-t_0)}$$

Similarly

$$P^t_X(k, t_0) = P_X(k - 1, t_0 - 1)e^{-\lambda(2-t_0)} \quad k = 2, 3, 4$$

Calculation of $P^*_X$ proceeds with similar arguments. If a path contributes to
$P^*_X(1, t_0)$ then it entered the interval in state 1 at $t = 1$ and did not change
state in the interval $(1, t_0)$. This means that its last stochastic state change
was to state 1 in the interval $(t_0 - 1, 1]$. Thus the path could have been in
any state at $t = t_0 - 1$, but it must have had at least one state change in the
interval $(t_0 - 1, 1)$ and the last change must have been to state 1. Thus:

$$P^*_X(1, t_0) = e^{-\lambda(t_0 - 1)}Q_0(t_0 - 1)e^{-\lambda(2-t_0)}Q_4(2-t_0) + P_X(2, t_0 - 1)e^{-\lambda(2-t_0)}Q_3(2-t_0) + P_X(3, t_0 - 1)e^{-\lambda(2-t_0)}Q_2(2-t_0) + P_X(4, t_0 - 1)e^{-\lambda(2-t_0)}Q_1(2-t_0)$$

$$= e^{-\lambda(t_0 - 1)}Q_0(t_0 - 1)Q_4(2-t_0) + Q_1(t_0 - 1)Q_3(2-t_0) + Q_2(t_0 - 1)Q_2(2-t_0) + Q_3(t_0 - 1)Q_1(2-t_0)$$

Notice in equation (25) the exponential decay acting as an amplitude is not
time dependent and the only time dependence comes through the $Q$ functions.
This is also true for the ‘creation’ contributions in (23) and (24).
The PMF’s for the remaining states follow using similar arguments. One obtains

\[ P^*_X(2, t_0) = e^{-\lambda}(Q_0(t_0 - 1)Q_1(2 - t_0) + Q_1(t_0 - 1)Q_4(2 - t_0) + Q_2(t_0 - 1)Q_3(2 - t_0) + Q_3(t_0 - 1)Q_2(2 - t_0)) \] (26)

\[ P^*_X(3, t_0) = e^{-\lambda}(Q_0(t_0 - 1)Q_2(2 - t_0) + Q_1(t_0 - 1)Q_1(2 - t_0) + Q_2(t_0 - 1)Q_4(2 - t_0) + Q_3(t_0 - 1)Q_3(2 - t_0)) \] (27)

and

\[ P^*_X(4, t_0) = e^{-\lambda}(Q_0(t_0 - 1)Q_3(2 - t_0) + Q_1(t_0 - 1)Q_2(2 - t_0) + Q_2(t_0 - 1)Q_1(2 - t_0) + Q_3(t_0 - 1)Q_4(2 - t_0)) \] (28)

With these expressions for the probability mass functions we can assemble the two wavefunctions by adding the creation and annihilation components. Thus we have, after some simplification:

\[ \psi_R(t_0) = P^\dagger_X(1, t) - P^\dagger_X(3, t) + P^*_X(1, t) - P^*_X(3, t) \]
\[ = e^{-\lambda}(\cos\lambda - (\sin\lambda(t_0 - 1) + \cos\lambda(t_0 - 1))) \] (29)

\[ \psi_L(t_0) = P^\dagger_X(2, t) - P^\dagger_X(4, t) + P^*_X(2, t) - P^*_X(4, t) \]
\[ = e^{-\lambda}(\sin\lambda - \sin\lambda(t_0 - 1)) \] (30)

Note that for \( \lambda = \pi/2 \) this becomes:

\[ \psi_R(t_0) = e^{-\pi/2}(\cos(\frac{\pi t_0}{2}) - \sin(\frac{\pi t_0}{2})) \] (31)

and

\[ \psi_L(t_0) = -e^{-\pi/2}(\cos(\frac{\pi t_0}{2}) + \sin(\frac{\pi t_0}{2})) \] (32)

Notice that \( \lim_{t_0 \to 1^+} \psi_R(t_0) = -e^{-\pi/2} = \psi_R(1) \) and \( \lim_{t_0 \to 1^+} \psi_L(t_0) = e^{-\pi/2} = \psi_R(1) \) so the wavefunctions in this region match the wavefunctions from the interaction region at \( t = 1 \).

**Subsequent deterministic regions \( t > 2 \)**

We could repeat the above calculation for subsequent time intervals; however, it is more instructive to note that all transitions are deterministic and all
state occupancies at time $t$ correspond to the occupation of the previous state at time $t - 1$. Thus if we consider a column vector of state occupation:

$$P_X(t) = (P_X(1, t), P_X(2, t), P_X(3, t), P_X(4, t))^T$$

(33)

Then for $t > 1$

$$P_X(t + 1) = (P_X(4, t), P_X(1, t), P_X(2, t), P_X(3, t))^T = EP$$

(34)

where

$$E = \begin{pmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}$$

(35)

Define the functions $\phi_R(t) = P_X(1, t) + P_X(3, t)$ and $\phi_L(t) = P_X(2, t) + P_X(4, t)$, then from the definitions of $\psi_R$ and $\psi_L$ we can write

$$\Phi(t) = \begin{pmatrix}
\psi_R(t), & \psi_L(t), & \phi_R(t), & \phi_L(t)
\end{pmatrix}^T$$

$$= \begin{pmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 0 & -1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{pmatrix} P_X(t) = R P_X(t)$$

where $R$ mediates the change of variables in (36). We have

$$\Phi(t + 1) = R P_X(t + 1)$$

$$= R EP X(t)$$

$$= R ER^{-1} \Phi(t)$$

(36)

Now

$$R ER^{-1} = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}$$

so

$$\Phi(t + 1) = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix} \Phi(t).$$

If we define $i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\Psi(t) = (\psi_R(t), \psi_L(t))^T$ then we see that

$$\Psi(t + 1) = i \Psi(t)$$

(37)

where $i^2 = -1$ illustrating our representation of the unit imaginary. The above may be generalized to

$$\Psi(t + n) = i^n \Psi(t) \quad n = 1, 2, \ldots$$

(38)
Note that equation (38) follows from the deterministic periodicity of our enumerative paths for $t > 1$. However let us take our expression for $\Psi(t)$ that we calculated for $1 < t_0 \leq 2$ and simply evaluate it at $t_0 + 1$. We have

$$\Psi(t_0) = e^{-\pi/2} (\cos(\pi t_0/2) - \sin(\pi t_0/2), \cos(\pi t_0/2) + \sin(\pi t_0/2))^T$$

so

$$\Psi(t_0 + 1) = e^{-\pi/2} (\cos(\pi (t_0 + 1)/2) - \sin(\pi (t_0 + 1)/2), \cos(\pi (t_0 + 1)/2) + \sin(\pi (t_0 + 1)/2))^T$$

$$= e^{-\pi/2} (- (\cos(\pi t_0/2) + \sin(\pi t_0/2)), \cos(\pi t_0/2) - \sin(\pi t_0/2))^T$$

$$= \imath \Psi(t_0)$$

so the expressions for the components of the wavefunction (31) and (32) that were calculated for the interval $[1, 2]$ are also valid on the interval $[2, 3]$ and in fact for $t > 1$ in general. The equation (39) is compared with a simulation of the stochastic model in Applets 11-13.

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References


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Figure 1: On the left is the graph of a single deterministic ‘entwined path’ in the $(x,t)$ plane. The colour indicates the direction of traversal, blue for traversal in the $+t$ direction, red for traversal in the $-t$ direction. The origin is at the bottom of the path and the lattice spacing is small on the scale of the figure. Notice that the crossing of the forward (blue) and backward (red) paths forms a chain of oriented areas, that is, a sequence of oriented areas concatenated along diagonal corners. The orientation switches from one rectangular area to the next and we can record this change in orientation using a single path that represents the right hand boundary of the areas. This is sketched to the right of the entwined path and we call such a path an enumerative path. The two density plots to the right of the enumerative path record the directed path density of the enumerative path on respectively the right and left light cone. The relation between the densities and the traversal of the entwined path is illustrated in Applet (1).
Figure 2: Two enumerative paths from a concatenated pair of entwined paths, where one envelope is displaced by 1/4 cycle from the other. They form a density with a square wave form. In the centre is the density resulting from the adolescent particles Eq(7) and on the right is the density for senescent particles.

Figure 3: A quartet of enumerative paths displaced so that they practically annihilate each other. The right-moving density is sketched and where the paths do not annihilate they leave an alternating density of discrete delta functions.
Figure 4: On the left is the graph of a single deterministic entwined pair with a stochastic first cycle. In the region $t < 1$ the outgoing path experiences two events from a Poisson process. The first event changes the particle’s direction, the second determines the crossing point for the return path. After the second event the particle suffers no more events in the interaction region and its path from then on is completely deterministic and regular until it returns to the interaction region where it completes its path back to the origin. Graph (B) in the middle of the figure is the resulting enumerative path for the pair to the left. The sketch in part (C) illustrates that the actual particle trajectory uses only four possible directions in spacetime which we label as states from one to four. In a quantum mechanical context, states three and four correspond to antiparticles. It is the statistical averaging of the four directions that lead to the Compton phase.
Figure 5: 10000 traversals with a stochastic first cycle. Only one component of the density is shown and the transient decay in the interaction region is displayed. The signal has six periods in the deterministic and periodic regime and the end of the signal exhibits a transient due to the artificial return of the particle after 6 deterministic cycles.

Figure 6: The periodic delta function of Eq(8) for a single quartet of EPs (cord) on the left. Sequential translations of cords are used to construct an approximation to $\sin(\pi x/2)$ in the right-hand figure.