

## Some Triality Structures

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**Abstract.** In this paper we write down our supertrality axioms, then we define the triality manifolds. Finally we define the cross product space. It is shown that  $V$  is a cross product space if and only if  $\dim(V) = 3$ .

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### 1. INTRODUCTION AND PRELIMINARIES

The role of Quantum Hall Effect and Zhang-Hu theory in physics is clear. The development of Q.H.E have attracted many attentions from different branches of mathematics such as Non-Commutative Geometry. In [4], the authors use triality structure to study Q.H.E. In this paper we first define supertrality axioms, hopefully to be useful to study super Q.H.E theory. Then we define triality manifolds, hopefully to be useful to study Q.H.E. On the other hand it is mathematically important in its own right.

In [4], the authors have introduced the triality axioms by the following:

If  $V$  be a real vector space equipped with the following structure:

-A grading by three element set  $S$ .

So  $V = \sum_{i \in S} V_i$ , where  $V_i$  is a subspace of  $V$  for each  $i \in S$  and for distinct  $i$  and  $j$  in  $S$ , we have  $V_i \cap V_j = \{0\}$ .

-A symmetric non-degenerate bilinear form  $g : V \times V \longrightarrow \mathbb{R}$ .

-A totally symmetric trilinear form  $m : V \times V \times V \longrightarrow \mathbb{R}$ .

-The multiplication map  $\mu : V \times V \longrightarrow V$  given by the formula:

$$g(x, \mu(v, w)) = m(v, w, x).$$

Then this structure is called triality if the following relations holds:

1- For distinct  $i$  and  $j$  in  $S$ ,  $V_i$  and  $V_j$  are orthogonal with respect to  $g$  and the restriction of  $g$  to each  $V_i$  be non-degenerate.

2-  $\mu$  is a graded multiplication: if  $v \in V_i$  and  $w \in V_j$ , then  $\mu(v, w) \in V_k$ , where  $i, j$  and  $k$  are distinct elements of  $S$ .

3-The restriction  $\mu$  to each  $V_i$  is trivial: for each  $i$  in  $S$  and for each  $v \in V_i$ ,  $\mu(v, v) = 0$ .

4- For each distinct  $i$  and  $j$  in  $S$  and for each  $v \in V_i$  and  $w \in V_j$ , we have the relation

$$\mu(v, \mu(v, w)) = g(v, v)w.$$

We start with some definitions and notations that will be used in the rest of the paper.

**Definition 1.1.** A superalgebra  $A = A_{\bar{0}} + A_{\bar{1}}$  is a  $\mathbb{Z}_2$ -graded algebra, that is,  $A_{\bar{i}} \cdot A_{\bar{j}} \subseteq A_{\bar{i}+\bar{j}}$ . so  $A_{\bar{0}}$  is the even part of  $A$  and is a subalgebra and  $A_{\bar{1}}$  is the odd part of  $A$  is a bimodule over  $A_{\bar{0}}$ .

**Definition 1.2.** Let  $\mathbb{B}$  be a super commutative algebra and  $M$  be a  $\mathbb{B}$ -bimodules, an even or odd morphism of  $\mathbb{B}$ -bimodules  $g : M \otimes M \longrightarrow \mathbb{B}$ , is called, respectively, an even or odd bilinear form, if  $g$  satisfying the following identities:

**a)**  $g$  is biadditive and homogenous.

**b)**  $g(ax, y) = (-1)^{\bar{g}\bar{a}}ag(x, y)$ , and  $g(x, ay) = g(x, y)a$ , for  $a \in \mathbb{B}$ ,  $x, y \in M$ .

we say that a form is symmetric if  $g(x, y) = (-1)^{\bar{x}\bar{y}}g(y, x)$ . We associate to an element  $x \in M$  the mapping  $x \rightarrow g(\cdot, x)$ , so we obtain a morphism  $M \rightarrow M^*$  of the same degree as  $g$ . Usually, we will denote this morphism by the same letter  $g$ . The form is said to be nondegenerate if  $g : M \longrightarrow M^*$  is an isomorphism.

## 2. Supertriality axioms

Let  $\mathbb{B}$  be a  $\mathbb{C}$ -vector superalgebra, consider a  $\mathbb{B}$ -bimodule  $M$ , equipped with the following structure:

.  $M = M_0 + M_1 + M_2$ , where  $M_j$  is a super submodule of  $M$  and for distinct  $i$  and  $j$ , We have  $M_i \cap M_j = \{0\}$ .

. A symmetric non-degenerate bilinear form  $g : M \otimes M \longrightarrow \mathbb{B}$ .

- . A totally symmetric trilinear form  $m : M \otimes M \otimes M \longrightarrow \mathbb{B}$ .
- . A multiplication map  $\mu : M \otimes M \longrightarrow M$  given by formula:

$$g(x, \mu(y, z)) = m(y, z, x).$$

We call this structure a supertriality provided that it satisfies the relations similar to triality.

We denote  $x^2, \langle x, y \rangle, xy$  and  $\langle x, y, z \rangle$  for  $g(x, x), g(x, y), \mu(x, y)$  and  $m(x, y, z)$ , respectively. Then we have the relations:

$$x(xy) = x^2y, \langle x, yz \rangle = \langle y, z, x \rangle .$$

In the following  $\alpha, \alpha', \alpha'', \dots$ , are generic elements in  $M_i$ ,  $\beta, \beta', \beta'', \dots$ , are generic elements in  $M_j$  and  $\gamma, \gamma', \gamma'', \dots$ , are generic elements in  $M_k$ , where  $i, j$  and  $k$  are distinct.

**Theorem 2.1.** *The following relations hold in a supertriality, for homogeneous elements  $\alpha, \beta, \gamma, \alpha', \beta'$ .*

**1-**  $\mu$  is symmetric.

$$\mathbf{2-} \alpha(\alpha'\beta) + \alpha'(\alpha\beta) = (1 + (-1)^{\tilde{\alpha}\tilde{\alpha}'} ) \langle \alpha, \alpha' \rangle \beta.$$

$$\mathbf{3-} \langle \alpha\beta, \alpha\beta \rangle = (-1)^{\tilde{\beta}(\tilde{\alpha}\tilde{\beta})} \beta^2 \alpha^2.$$

$$\mathbf{4-} \langle \alpha\beta, \alpha\beta' \rangle = (-1)^{\tilde{\beta}'(\tilde{\alpha}\tilde{\beta})} \langle \beta', \beta \rangle \alpha^2.$$

$$\mathbf{5-} (\alpha\beta)(\gamma\alpha) = (1 + (-1)^{\tilde{\gamma}(\tilde{\alpha}\tilde{\beta})} ) \langle \alpha, \beta, \gamma \rangle \alpha - (-1)^{\tilde{\alpha}(\tilde{\alpha}\tilde{\beta})} \gamma(\alpha^2\beta).$$

*proof:*1– For each  $x, y$  and  $z$  we have,  $\langle x, yz \rangle = \langle y, z, x \rangle = (-1)^{\tilde{y}\tilde{z}} \langle z, y, x \rangle = (-1)^{\tilde{y}\tilde{z}} \langle x, zy \rangle$ , so that  $\mu$  is symmetric.

2–We have,  $(\alpha + \alpha')((\alpha + \alpha')\beta) = (\alpha + \alpha')^2\beta$ , so that,

$$\alpha(\alpha\beta) + \alpha(\alpha'\beta) + \alpha'(\alpha\beta) + \alpha'(\alpha'\beta) = (\alpha^2 + \alpha'^2 + (1 + (-1)^{\tilde{\alpha}\tilde{\alpha}'}) \langle \alpha, \alpha' \rangle) \beta.$$

So **(2)** holds.

3—We have,  $\langle \alpha\beta, \alpha\beta \rangle = \langle \alpha, \beta, \alpha\beta \rangle = (-1)^{\tilde{\beta}(\tilde{\alpha\beta})} \langle \alpha, \alpha\beta, \beta \rangle = (-1)^{\tilde{\beta}(\tilde{\alpha\beta})} \langle \beta, \alpha(\alpha\beta) \rangle$ . So that **(3)** holds.

4—  $\langle \alpha\beta, \alpha\beta \rangle = \langle \alpha, \beta, \alpha\beta \rangle = (-1)^{\tilde{\beta}(\tilde{\alpha\beta})} \langle \alpha, \alpha\beta, \beta \rangle = (-1)^{\tilde{\beta}(\tilde{\alpha\beta})} \langle \beta, \alpha(\alpha\beta) \rangle$ . So that **(4)** holds.

5—By (2) we have,  $(1 + (-1)^{\tilde{\gamma}(\tilde{\alpha\beta})}) \langle \gamma, \alpha\beta \rangle \alpha = \gamma((\alpha\beta)\alpha) + (\alpha\beta)(\gamma\alpha) = (-1)^{\tilde{\alpha}(\alpha\beta)} \gamma(\alpha^2\beta) + (\alpha\beta)(\gamma\alpha)$ . So that **(5)** holds. ■

### 3. Triality manifolds

Consider a smooth manifold  $M$  equipped with the following structure:

- A symmetric, non-degenerate  $(0, 2)$  tensor field  $g \in \mathcal{T}_2^0(M)$  of constant index.
- A totally symmetric trilinear  $(0, 3)$  tensor field  $m \in \mathcal{T}_3^0(M)$ .
- A multiplication map  $\mu : \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$ .

We call  $(M, g, m, \mu)$  a triality manifold provided that the following relations hold:

- For each  $(X, Y, Z) \in \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M)$ , we have

$$g(X, \mu(Y, Z)) = m(Y, Z, X).$$

- For each  $p \in M$ , we have  $T_p(M) = T_p^0(M) + T_p^1(M) + T_p^2(M)$ , where  $T_p^i(M)$  is a subspace of  $T_p(M)$  and for distinct  $i$  and  $j$ , we have  $T_p^i(M) \cap T_p^j(M) = \{0\}$ .

- For each  $p \in M$  and for distinct  $i$  and  $j$ ,  $T_p^i(M)$  and  $T_p^j(M)$  are orthogonal with respect to  $g_p$  and restriction of  $g_p$  to each  $T_p^i(M)$  be non-degenerate.

- $\mu$  is a graded multiplication: if  $p \in M$ ,  $X \in T_p^i(M)$  and  $Y \in T_p^j(M)$  then  $\mu(X, Y) \in T_p^k(M)$ , where  $i, j$  and  $k$  are distinct.

- The restriction  $\mu$  to each  $T_p^i(M)$  is trivial: for each  $X \in T_p^i(M)$ ,  $\mu(X, X) = 0$ .

- For each distinct  $i$  and  $j$  and for each  $X \in T_p^i(M)$  and  $Y \in T_p^j(M)$ , we have the relation

$$\mu(X, \mu(X, Y)) = g(X, X)Y.$$

**Lemma 3.1.** *The multiplication map  $\mu : \mathcal{X}(M)^2 \longrightarrow \mathcal{X}(M)$  is a symmetric  $(1, 2)$  tensor field on  $M$ .*

**Proof.** For each  $(X, Y, Z) \in \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M)$  we have

$$g(X, \mu(Y, Z)) = m(Y, Z, X) = m(Z, Y, X) = g(x, \mu(Z, Y)).$$

So  $\mu$  is symmetric.

For each  $X, Y, Y'$  and  $Z \in \mathcal{X}(M)$  and  $f \in \mathcal{F}(M)$ , we have

$$g(X, \mu(fY + Y', Z)) = g(X, f\mu(Y, Z) + \mu(Y', Z))$$

So  $\mu$  is a  $\mathcal{F}(M)$ -multilinear function.

Let us introduce the following notations to simplify:

We denote  $X^2, \langle X, Y \rangle$  and  $\langle X, Y, Z \rangle$  for  $g(X, X), g(X, Y)$  and  $m(X, Y, Z)$ , respectively. In the following  $X, X', \dots$ , are generic elements in  $T_p^0(M)$ ,  $Y, Y', \dots$ , are generic elements in  $T_p^1(M)$  and  $Z, Z', \dots$ , are generic elements in  $T_p^2(M)$ . Then we have the relations:

$$\mu(X, \mu(X', Y)) + \mu(X', \mu(X, Y)) = 2 \langle X, X' \rangle Y.$$

$$\langle \mu(X, Y), \mu(X, Y) \rangle = X^2 Y^2, \quad \langle \mu(X, Y), \mu(X, Y') \rangle = X^2 \langle Y, Y' \rangle .$$

$$\langle \mu(X, Y), \mu(X', Y') \rangle + \langle \mu(X', Y), \mu(X, Y') \rangle = 2 \langle X, X' \rangle \langle Y, Y' \rangle .$$

$$\mu(\mu(X, Y), \mu(X, Z)) = 2 \langle X, Y, Z \rangle X - X^2 \mu(Y, Z).$$

#### 4. Cross product space

Consider a real vector space  $V$  equipped with the following structure:

- $V = \sum_{i=1}^3 V_i$ , where  $V_i$  is a nonzero subspace of  $V$  for each  $i$  and for distinct  $i$  and  $j$ , we have  $V_i \cap V_j = \{0\}$ .

-A symmetric, non-degenerate form  $g : V \times V \longrightarrow \mathbb{R}$ .

-A skew-symmetric, bilinear map  $\mu : V \times V \longrightarrow V$ .

We call this structure a cross product space provided that the following relations hold:

-For distinct  $i$  and  $j$ ,  $V_i$  and  $V_j$  are orthogonal with respect to  $g$  and the restriction of  $g$  to each  $V_i$  be non-degenerate.

- $\mu$  is graded multiplication: If  $v \in V_i$  and  $w \in V_j$ , then  $\mu(v, w) \in V_k$ , where  $i, j$  and  $k$  are distinct.

-The restriction  $\mu$  to each  $V_i$  is trivial: for each  $i$  and for each  $v, v' \in V_i$ ,  $\mu(v, v') = 0$ .

-For each distinct  $i$  and  $j$  and for  $v, \acute{v} \in V_i$  and  $w \in V_j$ , we have the relation,

$$\mu(\mu(v, w), \acute{v}) = g(v, \acute{v})w.$$

We abbreviate by writing  $a^2, \langle a, b \rangle$  and  $a \times b$  for  $g(a, a), g(a, b)$  and  $\mu(a, b)$ , respectively. Then we have the following Theorem.

**Theorem 4.1.** *If  $V$  be a cross product space then  $\dim(V) = 3$ .*

**Proof.** If  $e \in V_i$  such that  $e^2 \neq 0$  and  $\{f_1, \dots, f_m\}$  be a linear independent set of  $V_j$ , then  $\{e \times f_1, \dots, e \times f_m\}$  is a linear independent set of  $V_k$ , where  $i, j$  and  $k$  are distinct. So that  $\dim(V_1) = \dim(V_2) = \dim(V_3)$ . If  $\dim(V_1) > 1$ , since  $g|_{V_1}$  be non-degenerate we have orthogonal basis  $\{e_1, \dots, e_n\}$  of  $V_1$  such that  $n > 1$ . Consider  $f \in V_2$  such that  $f^2 \neq 0$ , then

$$(e_1 \times f) \times e_2 = \langle e_1, e_2 \rangle f = 0.$$

So that  $e_1 \times f = 0$ , this is a contradiction, therefor  $\dim(V_1) = 1$  and  $\dim(V) = 3$ .

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