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On New Laguere-Type of Functions Involving New Deformed Calculus

***Maslina Darus and Rabha W. Ibrahim**

School of Mathematical Sciences
Faculty of science and Technology
Universiti Kebangsaan Malaysia
Bangi 43600, Selangor Darul Ehsan, Malaysia
*maslina@ukm.my
rabhaibrahim@yahoo.com

Abstract: Generalized Laguerre polynomials are defined by using deformed calculus. Also, the Laguerre-type of some special functions are introduced with applications.

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*-corresponding author

1 Introduction.

Special functions appear in different frameworks. They are often used in combinatorial analysis [1], and even in statistics [2]. Moreover these polynomials have been applied even in many other contexts, such as the Blissard problem (see [1]), the representation of Lucas polynomials of the first and second kinds [3, 4], the representation formulas of Newton sum rules for polynomials zeros [5, 6], the recurrence relations for a class of Freud-type polynomials [7], the representation of symmetric functions of a countable set of numbers, generalizing the classical algebraic Newton-Girard formulas [8]. Consequently they were also used [9] in order to find reduction formulas for the orthogonal invariants of a strictly positive compact operator, deriving in a simple way the so-called Robert formulas [10].

In the present article we use the deformed calculus to define generalized Laguerre polynomials and other special functions. Moreover, explicit representation formulas for the deformed Laguerre-type derivative of a composite function are illustrated with applications.

2 Deformed Calculus.

Since special functions play important roles in mathematical physics, it is reasonable to imagine that some deformation of the ordinary special functions based on the new deformed calculus. The new deformed calculus was developed in [11].

For requirement, a new kind of deformed number system which is defined by

$$[n] = n + \frac{p-1}{2}(1-(-)^n). \tag{2.1}$$

Obviously, $[2k] = 2k, [2k+1] = 2k+p$ for any integer k and when $p \to 1, [n] \to n$. A new derivative operator D can be defined which acts on function $f(x)$ as

$$\begin{aligned} Df(x) \equiv \frac{D}{Dx}f(x) &= \frac{d}{dx}f(x) + \frac{p-1}{2x}(1-R)f(x) \\ &= df(x) + \frac{p-1}{2x}(f(x)-f(-x)), \end{aligned} \tag{2.2}$$

where $df = \frac{d}{dx}f$ and R satisfies $Rf(x) = f(-x)$. Equation (2) implies that D acts on an even function $f_e(-x) = f_e(x)$ as the ordinary derivative $Df_e(x) = df_e(x)$, and D acts on an odd function $f_o(-x) = -f_o(x)$ leads to $Df_o(x) = df_o(x) + \frac{p-1}{x}f_o(x)$. For $p = 1$, D reduces to the ordinary derivative operator d. Generalization of the ordinary differential relation $dx^n = nx^{n-1}$ reads

$$Dx^n = [n]x^{n-1} \tag{2.3}$$

and $E(x)$ defined by

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]!} \tag{2.4}$$

with $DE(x) = E(x)$, where $[n]! = [n][n-1]...[1]$ and $[0]! \equiv 1$. Thus the $E(x)$ is a deformation of the ordinary exponential function e^x in our case and it will reduce to e^x when $p \to 1$.

Deformed derivative operator D also leads to a new deformed integration (indefinite integral) which formally written as

$$\begin{aligned} I(x) := \int DxF(x) &= \sum_{n=0}^{\infty}(-)^n\Big(\int dx\frac{p-1}{2x}(1-R)\Big)^n \int dxF(x) \\ &= \int dxF(x) - \int dx\frac{p-1}{2x}(1-R)\int dxF(x) \\ &+ \Big(\int dx\frac{p-1}{2x}(1-R)\Big)^2 \int dxF(x) - \dots . \end{aligned} \tag{2.5}$$

And the definite integral is defined as

$$\begin{aligned}\int_a^b DxF(x) &= \int_a^b dx \sum_{n=0}^{\infty} (-)^n \Big(\frac{p-1}{2x}(1-R)\int_a^x dx\Big)^n F(x) \\ &= \int_a^b dxF(x) - \int_a^b dx\frac{p-1}{2x}(1-R)\int_a^x dxF(x) \\ &+ \int_a^b dx\frac{p-1}{2x}(1-R)\int_a^x \frac{p-1}{2x}(1-R)\int_a^x dxF(x) - \dots .\end{aligned} \tag{2.6}$$

Corresponding to (2.3), yields

$$\int Dxx^n = \frac{x^{n+1}}{[n+1]} + c,$$

where c is the integration constant. Finally, the usual Leibnitz rule also works for the deformed operator D

$$D(fg) = (Df)g + f(Dg), \tag{2.7}$$

where either $f(x)$ or $g(x)$ is an even function of x. If either F(x) or G(x) is an even function of x, one has a formula of integration by parts from Eq.(2.7)

$$\int_a^b Dx\frac{DF}{Dx}G = FG|_a^b - \int_a^b DxF\frac{DG}{Dx}. \tag{2.8}$$

3 Deformed polynomials.

In this section, we define deformed Laguerre polynomials and study some of their properties.

Definition 3.1. The Laguerre polynomials take the form

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}e^{-x}x^n, \;\; L_0 = 1, \;\; n \in \mathbb{N}_0.$$

For a complex variable $z \in \mathbb{C}$ we define deformed Laguerre polynomials by using deformed calculus in Section 2.

Definition 3.2. The deformed Laguerre polynomials take the form

$$\mathcal{L}_n(z) = \frac{E(z)}{[n]!}D^n z^n E(-z), \;\; \forall\, z \in \mathbb{C}. \tag{3.9}$$

Then we have the following recurrence relations

Theorem 3.1. Let $n \in \mathbb{N}, z \in \mathbb{C}$. Then the generalized Laguerre polynomials $\mathcal{L}_n(z)$ of order n satisfy

1. $\mathcal{L}_0(z) = 1$.
2. $zD\mathcal{L}_n(z) = [n]\mathcal{L}_n(z) - n\mathcal{L}_{n-1}(z)$.
3. $[n+1]\mathcal{L}_{n+1}(z) = \Big(\dfrac{2n+1-z}{n+1}\Big)[n+1]\mathcal{L}_n(z) - \Big(\dfrac{n}{n+1}\Big)[n+1]\mathcal{L}_{n-1}(z)$.

Proof. Following the steps below:

1. From Eq. (2.4).
2. By (2.2), we receive

$$\begin{aligned} zD\mathcal{L}_n(z) &= z\frac{d}{dz}\mathcal{L}_n(z) + \frac{p-1}{2}(1-R)\mathcal{L}_n(z) \\ &= (n + \frac{p-1}{2}(1-R))\mathcal{L}_n(z) - n\mathcal{L}_{n-1}(z) \\ &= [n]\mathcal{L}_n(z) - n\mathcal{L}_{n-1}(z). \end{aligned} \tag{3.10}$$

3.

$$\begin{aligned} [n+1]\mathcal{L}_{n+1}(z) &= (n+1)\mathcal{L}_{n+1}(z) + \frac{p-1}{2}(1-(-1)^{n+1})\mathcal{L}_{n+1}(z) \\ &= (2n+1-z)\mathcal{L}_n(z) - n\mathcal{L}_{n-1}(z) \\ &+ \frac{p-1}{2}(1-(-1)^{n+1})\{\frac{(2n+1-z)}{n+1}\mathcal{L}_n(z) - \frac{n}{n+1}\mathcal{L}_{n-1}(z)\} \\ &= (2n+1-z)\{1 + \frac{p-1}{2(1+n)}(1-(-1)^{n+1})\}\mathcal{L}_n(z) \\ &- n\{1 + \frac{p-1}{2(1+n)}(1-(-1)^{n+1})\}\mathcal{L}_{n-1}(z) \\ &= \{1 + \frac{p-1}{2(1+n)}(1-(-1)^{n+1})\}\{(2n+1-z)\mathcal{L}_n(z) - n\mathcal{L}_{n-1}(z)\} \\ &= \Big(\frac{2n+1-z}{n+1}\Big)[n+1]\mathcal{L}_n(z) - \Big(\frac{n}{n+1}\Big)[n+1]\mathcal{L}_{n-1}(z). \end{aligned}$$

Theorem 3.2. Let $n \in \mathbb{N}, z \in \mathbb{C}$. Then the deformed Laguerre polynomials $\mathcal{L}_n(z)$ of order n are solutions of deformed Laguerre equation based on the deformed derivative operator

$$zD^2\mathcal{L}_n(z) + (p - [n] + n - z)D\mathcal{L}_n(z) + [n]\mathcal{L}_n(z) = 0. \tag{3.11}$$

Proof. Differentiating (3.10), applying (2.7) and using the fact that $Dz = p$, we receive (3.11).

Another generalization holds for the deformed Laguerre polynomials as follows:

Definition 3.3. The deformed generalized Laguerre polynomials take the form

$$\mathcal{L}_n^k(z) = \frac{z^{-k}E(z)}{[n]!} D^n z^{n+k} E(-z), \ \ \forall\, z \in \mathbb{C}. \tag{3.12}$$

Then we have the following recurrence relations:

Theorem 3.3. Let $n \in \mathbb{N}, z \in \mathbb{C}$. Then the deformed generalized Laguerre polynomials $\mathcal{L}_n^k(z)$ satisfy

1. $\mathcal{L}_n^0(z) = \mathcal{L}_n(z)$.
2. $\mathcal{L}_0^0(z) = 1$.
3. $D\mathcal{L}_n^k(z) = \frac{1}{z}\big([n]\mathcal{L}_n^k(z) - (n+k)\mathcal{L}_{n-1}^k(z)\big)$.

Proof. Following the steps below:

1. From the Definition (3.3).
2. From Eq. (2.4).
3. By applying (2.2), yields

$$\begin{aligned} zD\mathcal{L}_n^k(z) &= z\frac{d}{dz}\mathcal{L}_n^k(z) + \frac{p-1}{2}(1-R)\mathcal{L}_n^k(z) \\ &= (n + \frac{p-1}{2}(1-R))\mathcal{L}_n^k(z) - (n+k)\mathcal{L}_{n-1}^k(z) \\ &= [n]\mathcal{L}_n^k(z) - (n+k)\mathcal{L}_{n-1}^k(z). \end{aligned} \tag{3.13}$$

Consequently, we have the following result.

Theorem 3.4. Let $n \in \mathbb{N}, z \in \mathbb{C}$. Then the generalized deformed Laguerre polynomials $\mathcal{L}_n^k(z)$ of order n are solutions of deformed Laguerre equation

$$zD^2\mathcal{L}_n^k(z) + (k + p - [n] + n - z)D\mathcal{L}_n^k(z) + [n]\mathcal{L}_n^k(z) = 0. \tag{3.14}$$

Proof. Differentiating (3.13), applying (2.7) we receive (3.14).

4 Deformed Laguerre-type functions.

In this section, the deformed Laguerre-type derivatives introduced and connected with a deformed differential isomorphism denoted by the symbol $\mathcal{D}_z^{-1}$ acting onto the space $\mathcal{A}$ of analytic functions of the z variable as follows:

$$\mathcal{D}_z^{-1} f(z) = \int_0^z f(\zeta) D\zeta.$$

In general,

$$\mathcal{D}_z^{-n} f(z) := \frac{1}{[n-1]!}\int_0^z f(\zeta)(z-\zeta)^{n-1} D\zeta.$$

Thus we have

$$\mathbf{F}(z^n) := \mathcal{D}_z^{-n}(1) = \frac{1}{[n-1]!}\int_0^z (z-\zeta)^{n-1} D\zeta = \frac{z^n}{[n]!}.$$

According to this concept, the deformed exponential operator $E(z)$ is transformed into the first deformed Laguerre-type exponential

$$\mathbf{F}(E(z)) = \sum_{n=0}^{\infty} \frac{\mathbf{F}(z^n)}{[n]!} = \sum_{n=0}^{\infty} \frac{z^n}{([n]!)^2} = E_1(z)$$

which is called deformed $\mathcal{L}-$exponential functions, or shortly deformed $\mathcal{L}-$exponentials. So for order m we receive

$$\mathbf{F}^m(E(z)) = \sum_{n=0}^{\infty} \frac{\mathbf{F}(z^n)}{([n]!)^m} = \sum_{n=0}^{\infty} \frac{z^n}{([n]!)^{m+1}} = E_m(z)$$

For the deformed hypergeometric function

$${}_qF_p[\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_p; z] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n ... (\alpha_q)_n}{(\beta_1)_n ... (\beta_p)_n} \frac{z^n}{[n]!}$$

where $(a)_n$ is the deformed Pochhammer symbol defined by

$$(a)_n = \begin{cases} 1, & n = 0 \\ 1[a][a+1]...[a+n-1], & n = \{1, 2, ...\}. \end{cases}$$

we can apply the deformed Laguerre derivative to find the deformed Laguerre type hypergeometric function

$$\begin{aligned}\mathbf{F}({}_qF_p[\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_p; z]) &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n ... (\alpha_q)_n}{(\beta_1)_n ... (\beta_p)_n} \frac{\mathbf{F}(z^n)}{[n]!} \\ &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n ... (\alpha_q)_n}{(\beta_1)_n ... (\beta_p)_n} \frac{z^n}{([n]!)^2} \\ &=_q F_p^1[\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_p; z].\end{aligned}$$

For m order we receive

$$\begin{aligned}\mathbf{F}^m({}_qF_p[\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_p; z]) &= \sum_{n=0}^{\infty} \Big[\frac{(\alpha_1)_n ... (\alpha_q)_n}{(\beta_1)_n ... (\beta_p)_n}\Big]^m \frac{\mathbf{F}(z^n)}{([n]!)^m} \\ &= \sum_{n=0}^{\infty} \Big[\frac{(\alpha_1)_n ... (\alpha_q)_n}{(\beta_1)_n ... (\beta_p)_n}\Big]^m \frac{z^n}{([n]!)^{m+1}} \\ &=_q F_p^m[\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_p; z],\end{aligned}$$

the deformed Laguerre-type hypergeometric function of order m.

Next we introduce the deformed Laguerre-type Fox-Wright function. For complex parameters

$$\alpha_1, ..., \alpha_q \quad (\frac{\alpha_j}{A_j} \neq 0, -1, -2, ... \; ; j = 1, ..., q)$$

and

$$\beta_1, ..., \beta_p \quad (\frac{\beta_j}{B_j} \neq 0, -1, -2, ... \; ; j = 1, ..., p),$$

the deformed Fox-Wright generalization ${}_q\Psi_p[z]$ of the deformed hypergeometric ${}_qF_p$ function by

$$\begin{aligned}
{}_q\Psi_p \left[\begin{array}{ll} (\alpha_1, A_1), ..., (\alpha_q, A_q); & \\ & z \\ (\beta_1, B_1), ..., (\beta_p, B_p); & \end{array} \right] &=_q \Psi_p[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,p}; z] \\
&:= \sum_{n=0}^{\infty} \frac{\Gamma[\alpha_1 + nA_1]...\Gamma[\alpha_q + nA_q]}{\Gamma[\beta_1 + nB_1]...\Gamma[\beta_p + nB_p]} \frac{z^n}{[n]!} \\
&= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{q} \Gamma[\alpha_j + nA_j]}{\prod_{j=1}^{p} \Gamma[\beta_j + nB_j]} \frac{z^n}{[n]!}
\end{aligned}$$

where $A_j > 0$ for all $j = 1, ..., q$, $B_j > 0$ for all $j = 1, ..., p$ and $1+\sum_{j=1}^{p} B_j - \sum_{j=1}^{q} A_j \geq 0$ for suitable values $|z|$. The deformed Laguerre-type ${}_q\Psi_p[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,p}; z]$

$$\begin{aligned}
\mathbf{F}({}_q\Psi_p[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,p}; z]) &= \sum_{n=0}^{\infty} \frac{\Gamma[\alpha_1 + nA_1]...\Gamma[\alpha_q + nA_q]}{\Gamma[\beta_1 + nB_1]...\Gamma[\beta_p + nB_p]} \frac{\mathbf{F}(z^n)}{[n]!} \\
&= \sum_{n=0}^{\infty} \frac{\Gamma[(\alpha_1 + nA_1]...\Gamma[(\alpha_q + nA_q]}{\Gamma[(\beta_1 + nB_1]...\Gamma[(\beta_p + nB_p]} \frac{z^n}{([n]!)^2} \\
&=_q \Psi_p^1[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,p}; z],
\end{aligned}$$

where $\Gamma[n] := [n-1]!$ denoted the deformed Gamma function. For m order we receive

$$\begin{aligned}
\mathbf{F}^m({}_q\Psi_p[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,p}; z]) &= \sum_{n=0}^{\infty} [\frac{\Gamma[\alpha_1 + nA_1]...\Gamma[\alpha_q + nA_q]}{\Gamma[\beta_1 + nB_1]...\Gamma[\beta_p + nB_p]}]^m \frac{\mathbf{F}(z^n)}{([n]!)^m} \\
&= \sum_{n=0}^{\infty} [\frac{\Gamma[\alpha_1 + nA_1]...\Gamma[\alpha_q + nA_q]}{\Gamma[\beta_1 + nB_1]...\Gamma[\beta_p + nB_p]}]^m \frac{z^n}{([n]!)^{m+1}} \\
&=_q \Psi_p^m[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,p}; z],
\end{aligned}$$

the deformed Laguerre-type Fox-Wright function of order m. From the above examples of analytic functions, we can define new classes in $\mathcal{A}$ and can be very useful in the theory of univalent functions.

5 Deformed heat equation.

The heat equation is of fundamental importance in diverse scientific fields. In mathematics, it is the prototypical parabolic partial differential equation. In statistics, the heat equation is connected with the study of Brownian motion via the Fokker-Planck equation. The diffusion equation, a more general version of the heat equation, arises in connection with the study of chemical diffusion and other related processes.

In term of deformed calculus we introduce the generalized heat equation as follows

$$\frac{Du}{Dt} = K\frac{Du^2}{D^2x} \tag{5.15}$$

where u is the temperature that depends on position (x) and time (t). And K is a constant refers to thermal diffusivity and the operator D is a deformed derivative (2.2).

By using the definition of the operator D, equation (5.15) reduces to form

$$\frac{du}{dt} = K\frac{d^2u}{dx^2} + \rho(x)\frac{du}{dx} + g(t,x)u, \tag{5.16}$$

where $\rho(x) := \frac{K(p-1)(1-R)}{x}$ and $g(t,x) := \frac{(p-1)(1-R)(Kt+x^2)}{tx^2}$. It is clear that when $\rho(x) = g(t,x) = 0$ (this holds at $p = 1$) we obtain the well known heat equation which has a solution of the form

$$u(t,x) = \sum_{s=1}^{\infty} A_s \exp(-s^2Kt)\sin sx. \tag{5.17}$$

(see [12]). Equation (5.16) can be written in more general form as follows:

$$\frac{du}{dt} = K\frac{d^2u}{dx^2} + G(t,x,u,u_x), \tag{5.18}$$

which has many applications in intermediate physical process; convection-diffusion and transport-diffusion processes.

Example 5.1. As an example of the application of deformed calculus and deformed special functions in applied mathematics, we shall consider the problem of determining the temperature in a slab of homogeneous material bounded by the planes $x = o$ and $x = \pi$; having an initial temperature $u = f(x)$, varying only with the distances from the faces and with its two faces kept at zero temperature. The formula for the temperature u at any instant and at all points of the slab is to be determined. In the problem, it is clear that the temperature function is of the variables x and t only. Hence at each interior point this function $u(x,t)$ must satisfy the heat equation for one dimensional form. In term of deformed calculus, we introduce the deformed heat equation as follows

$$\frac{Du}{Dt} = K\frac{Du^2}{D^2x} \tag{5.19}$$

subject to the conditions

$$\begin{aligned} &u(0,t) = 0, \\ &u(\pi,t) = 0, \ \ (t > 0) \\ &u(x,0) = f(x), \ \ (0 < x < \pi). \end{aligned} \tag{5.20}$$

Assume that $f(x) = e^{2imx} cos2nx$ and $p \to 1$. The solution of the boundary value problem (5.19-5.20) takes the form

$$u(t,x) = \begin{cases} 0, \ \ \text{if } \ m \neq n; \\ \sum_{s=1}^{\infty} \frac{\pi}{4} s \exp(-s^2 Kt) \sin sx, \text{ if } \ m = n \neq 0 \\ \sum_{s=1}^{\infty} \frac{\pi}{2} s \exp(-s^2 Kt) \sin sx, \text{ if } \ m = n = 0. \end{cases}$$

6 Conclusion.

By using new kind of deformed calculus, a generalization of different kind of special functions (Laguerre, hypergeometric, Fox-Wright) is defined and studied. This leads to define new classes of analytic functions in $\mathcal{A}$ which are important in the theory of univalent functions. Therefore, the deformed heat equation is of general character and hence may encompass several cases of interest in intermediate physical process.

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