Three Lorentz Transformations

Considering Two Rotations

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Abstract

A formulation for combination of three Lorentz transformations considering two rotations has been developed in this paper. This procedure physically means the composition of three linear velocities in different directions resulting in a single velocity. The same process has been applied to a particle or body possessing three simultaneous rotational motions having mutual effects. This leads to the assumption that a body may possess three simultaneous superimposed spins.

Key words: superimposed rotational motions, superimposed spins.

1. Introduction

One of the basic questions in Lorentz transformation is velocity addition. According to Einstein’s relativistic theorem, we have the well known formula of co-linear velocity addition[1], \( u = \frac{u' + v}{1 + u'v/c^2} \). Møller extended this approach to develop the velocity addition formula for combining two velocities in different directions in planar motion [2]. Chandrau Iyer and G.M Probhhu describe the \( LRL \) transformation for composition of two Lorentz boosts in different directions in plane [3]. This means the composition of two velocities \( u \) and \( v \) resulting in a single velocity \( w \). In this paper we first develop the \( LRLRL \) transformations yielding a

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single resultant Lorentz boost through a proper way. We extend this to compose three superimposed motions in different directions into a resultant velocity and following this constructive method we show that a body or particle may possess three simultaneous superimposed spins.

2. LRLRL Transformations

In this section we want to derive the matrix for a conventional Lorenz transformation $L_{xw}$ followed by a planar rotation of the $XY$ plane ($R_\theta$) and then followed by another conventional Lorenz transformation $L_{xy}$ which is again followed by a planar rotation of the $XY$ plane ($R_\psi$) and then by another conventional Lorenz transformation $L_{sw}$. For clarity, we may visualize six inertial frames $S, S_1, S_2, S_3, S_4$ and $S_5$ in three dimensional space. Frames $S$ and $S_1$ have both their co-ordinate axes aligned and $S_1$ is moving at a velocity $u$ along $X_1$ axis as observed by $S$. The inertial frame $S_1$ has another co-ordinate reference frame $S_2$, where $X_2$ axis of $S_2$, are rotated by an angle $\theta$ counter clockwise with respect to $S_1$ on $X_1Y_1$ plane. Frames $S_2$ and $S_3$ have both their co-ordinate axes aligned and $S_3$ is moving at a velocity $v$ along $X_3$ axis as observed by $S_2$. The inertial frame $S_3$ has another coordinate reference frame $S_4$ where $X_4$ axis of $S_4$ are rotated by an angle $\psi$ counter clockwise with respect to $S_3$ on $X_3Y_3$ plane. Frames $S_4$ and $S_5$ have both their co-ordinate axes aligned and $S_5$ is moving at a velocity $w$ along $X_5$ axis as observed by $S_4$. Then the transformation of co-ordinates of an event from frame $S$ to frame $S_5$ is given by $4 \times 4$ order matrix $M_{ij}$ which is equal to the matrix product $L_{xw} R_{xz(\psi)} L_{xv} R_{xy(\theta)} L_{wu}$. The matrix $L_{xw}, R_{xy(\theta)}, L_{sv}, R_{xz(\psi)}$ and $L_{sw}$ are as specified in (1), (2), (3), (4) and (5) respectively.
Lorentz transformations

\[
\begin{pmatrix}
1 & 2 & 1 \\
1 & 2 & 1 \\
1 & 2 & 1 \\
1 & 2 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 2 & 1 \\
1 & 2 & 1 \\
1 & 2 & 1 \\
1 & 2 & 1 2
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 0 & 0 & -u \gamma_u \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\frac{u \gamma_u}{c^2} & 0 & 0 & \gamma_u
\end{pmatrix}
\]

\[
\begin{pmatrix}
x_1 \\
y_1 \\
z_1 \\
t_1
\end{pmatrix} =
\begin{pmatrix}
\gamma_u & 0 & 0 & -u \gamma_u \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\frac{u \gamma_u}{c^2} & 0 & 0 & \gamma_u
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z \\
t
\end{pmatrix} (1),
\begin{pmatrix}
x_2 \\
y_2 \\
z_2 \\
t_2
\end{pmatrix} =
\begin{pmatrix}
\cos \theta & \sin \theta & 0 & 0 \\
-\sin \theta & \cos \theta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
y_1 \\
z_1 \\
t_1
\end{pmatrix} (2)
\]

\[
\begin{pmatrix}
x_3 \\
y_3 \\
z_3 \\
t_3
\end{pmatrix} =
\begin{pmatrix}
\gamma_v & 0 & 0 & -v \gamma_v \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\frac{v \gamma_v}{c^2} & 0 & 0 & \gamma_v
\end{pmatrix}
\begin{pmatrix}
x_2 \\
y_2 \\
z_2 \\
t_2
\end{pmatrix} (3),
\begin{pmatrix}
x_4 \\
y_4 \\
z_4 \\
t_4
\end{pmatrix} =
\begin{pmatrix}
\cos \psi & 0 & \sin \psi & 0 \\
0 & 1 & 0 & 0 \\
-\sin \psi & 0 & \cos \psi & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x_3 \\
y_3 \\
z_3 \\
t_3
\end{pmatrix} (4)
\]

\[
\begin{pmatrix}
x_5 \\
y_5 \\
z_5 \\
t_5
\end{pmatrix} =
\begin{pmatrix}
\gamma_w & 0 & 0 & -w \gamma_w \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\frac{w \gamma_w}{c^2} & 0 & 0 & \gamma_w
\end{pmatrix}
\begin{pmatrix}
x_4 \\
y_4 \\
z_4 \\
t_4
\end{pmatrix} (5)
\]

The inverse of this operation is given by

\[
N_{ij} = L_{x(-u)} R_{xy(-\theta)} L_{x(-v)} R_{xz(-\psi)} L_{x(-w)}
\]

Matrices \(M_{ij}\) and \(N_{ij}\) turn out as

\[
M_{ij} = L_{sw} R_{xz(\psi)} L_{xy(\theta)} L_{sx(\psi)} L_{sw} =
\]

\[
\begin{pmatrix}
\gamma_w & 0 & 0 & -w \gamma_w \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\frac{w \gamma_w}{c^2} & 0 & 0 & \gamma_w
\end{pmatrix}
\begin{pmatrix}
\cos \psi & 0 & \sin \psi & 0 \\
0 & 1 & 0 & 0 \\
-\sin \psi & 0 & \cos \psi & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\gamma_v & 0 & 0 & -v \gamma_v \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\frac{v \gamma_v}{c^2} & 0 & 0 & \gamma_v
\end{pmatrix}
\]

\[
\begin{pmatrix}
\cos \theta & \sin \theta & 0 & 0 \\
-\sin \theta & \cos \theta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\gamma_u & 0 & 0 & -u \gamma_u \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\frac{u \gamma_u}{c^2} & 0 & 0 & \gamma_u
\end{pmatrix}
\]
Similarly \( N_{ij} = L_{x(-u)} R_{xy(-\theta)} L_{x(-v)} R_{xz(-\psi)} L_{x(-w)} \) =

\[
\begin{pmatrix}
\gamma_u & 0 & 0 & u\gamma_u \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\frac{u\gamma_u}{c^2} & 0 & 0 & \gamma_u
\end{pmatrix}
\begin{pmatrix}
\cos \theta & -\sin \theta & 0 & 0 \\
\sin \theta & \cos \theta & 0 & 0 \\
0 & 0 & 1 & 0 \\
\frac{v\gamma_u}{c^2} & 0 & 0 & \gamma_u
\end{pmatrix}
\begin{pmatrix}
\gamma_v & 0 & 0 & v\gamma_v \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\frac{w\gamma_v}{c^2} & 0 & 0 & \gamma_v
\end{pmatrix}
\begin{pmatrix}
\cos \psi & 0 & -\sin \psi & 0 \\
0 & 1 & 0 & 0 \\
\sin \psi & 0 & \cos \psi & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\gamma_w & 0 & 0 & w\gamma_w \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\frac{w\gamma_w}{c^2} & 0 & 0 & \gamma_w
\end{pmatrix}
\]
3. Resultant velocity due to composition of three velocities

For composition of three velocities we may visualize six inertial frames as discussed in section-2 where, the co-ordinate of an event at any point on $S_5$ with respect to $S$ would be written as

$$ H = N_{ij} K $$

where, $H = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$, $K = \begin{pmatrix} x_5 \\ y_5 \\ z_5 \\ t_5 \end{pmatrix}$

Equation (8) gives the relations as
From spherical coordinate system one can also write

\[
\frac{dx}{dt_5} = w\sin(90^\circ - \psi)\cos \theta = w\cos \theta \cos \psi \\
\frac{dy}{dt_5} = w\sin(90^\circ - \psi)\sin \theta = w\sin \theta \cos \psi \\
\frac{dz}{dt_5} = w\cos(90^\circ - \psi) = w\sin \psi
\]

Following equations (9) and (10) we could find out the resultant velocity 'G' of the event of S_i as observed by S_j whose magnitude also can be evaluated by considering the motion of the origin of S_j (x_j = 0, y_j = 0, z_j = 0) as

\[
G = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}
\]

This may be written as

\[
G = \sqrt{P^2 + Q^2 + R^2} \\
\frac{T}{T}
\]

where,

\[
P = N_{11}w\cos \theta \cos \psi + N_{12}w\sin \theta \cos \psi + N_{13}w\sin \psi + N_{14} \\
Q = N_{21}w\cos \theta \cos \psi + N_{22}w\sin \theta \cos \psi + N_{23}w\sin \psi + N_{24} \\
R = N_{31}w\cos \theta \cos \psi + N_{32}w\sin \theta \cos \psi + N_{33}w\sin \psi + N_{34} \\
T = N_{41}w\cos \theta \cos \psi + N_{42}w\sin \theta \cos \psi + N_{43}w\sin \psi + N_{44}
\]

When \( \theta = \psi = 90^\circ \) then the resultant velocity G would be given by

\[
G' = (u^2 + v^2 + w^2 + \frac{u^2v^2w^2}{c^4} - \frac{u^2v^2}{c^2} - \frac{v^2w^2}{c^2} - \frac{w^2u^2}{c^2})^{\frac{1}{2}}
\]

\[
(13)
\]
4. Resultant velocity due to three rotational motions

From the above discussion it is seen that $u$, $v$, and $w$ are the velocities in three different directions where, angle between $u$ and $v$ is $\theta$, between $v$ and $w$ is $\psi$ and that between $w$ and $u$ is $\phi$ which depends on $\theta$ and $\psi$. In that sense for composition or addition of three simultaneous superimposed rotational motions we can consider the velocity addition formula (12). For clarity we may visualize six inertial frames as discussed in section-2, but $S_1$, $S_3$ and $S_5$ don’t move with rectilinear motion. They move only with angular velocity $\omega_1$, $\omega_2$ and $\omega_3$ respectively, about $X_1$, $X_3$ and $X_5$ axes as observed by $S$ and others (i.e. $S_2$ and $S_4$) be same as discussed in section-2, When origins of frames are same with respect to $S$, then the resultant velocity of an event of $S_5$ will be same as $G$ in (12) as observed by $S$ where, $\vec{u} = \omega_1 \times \vec{r}$, $\vec{v} = \omega_2 \times \vec{r}$ and $\vec{w} = \omega_3 \times \vec{r}$ become separately relativistic velocities and $\vec{r}$ is the position vector of the event in $S_5$. It is to be mentioned that if a body or particle be present at the origin of $S_5$ then it will possess three simultaneous superimposed spins as observed by $S$.

5. Conclusions

The above derivations and discussions reveal the field of relativistic effects on a body due to three motions. The expression for resultant velocity shows that it is dependent upon all the linear velocities arising from the rotations (viz.equation (12))

References

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