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Anisotropic Cosmological Models with Conformally Invariant Scalar Fields and Constant Deceleration Parameter

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Abstract

Einstein's field equations for a spatially homogeneous and anisotropic Bianchi type-I space-time in the presence of a conformally invariant scalar field are discussed and their solutions are obtained by applying the law of variation of Hubble parameter, which yields a constant value of the deceleration parameter. The law generates power-law and exponential forms of the average scale factor in terms of cosmic time. Exact solutions of the field equations that correspond to singular and non-singular cosmological models are presented by using the respective forms of the average scale factor.The kinematical properties of the models are discussed. It is observed that the solutions are consistent with the observations of the present-day universe.

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1. INTRODUCTION

General relativity couples gravity with all fields. It is held that the large-range forces are produced by scalar fields. Scalar-tensor theories of gravitation have become a focal point of interest in many areas of gravitational physics and cosmology. They provide that most natural generalizations of general relativity and provide a convenient set of representations for the observational limits on possible deviations from general relativity. The study of zero-rest-mass scalars fields in general relativity has been initiated to provide an understanding of the nature of space-time and gravitational field associated with neutral elementary particles of zero spin. Bergmann and Leipnik [1] studied solutions of Einstein's field equations in static, spherically symmetric space-time with an energy-momentum tensor generated from a neutral massless scalar fields. Callan et al.[2] proposed a theory of gravitation in which the renormalized energy-momentum tensor defines the same four-momentum and Lerentz generators as the conventional tensor. The theory meets all the experimental tests that have been applied to general relativity. Frϕyland [3] studied zero-mass scalars fields coupled to the gravitational field in the static, spherically symmetric case, is completely solve for a traceless energy-momentum tensor.

Accioly et al.[4] obtained solutions of field equations to the conformally invariant scalar field with tracefree energy-momentum tensor as source in Bianchi type-I space-time. Innaiah and Reddy [5] have discussed a flat Robertson-Walker-type solution treating the conformally invariant scalar fieldas the matter field. Venkateswarlu and Reddy [6] have obtained Bianchi type-II, VIII and IX cosmological solutions in conformally-invariant scalar field with trace-free electromagnetic energy-momentum tensor. Maharaj and Beeaham [7] have discussed Robertson-Walker-type universe with conformally-invariant scalar field to be independent of time. Shri Ram [8,9] has obtained a exact solutions of the Einstein equations corresponding to a conformally invariant scalar field with trac-free energy-momentum tensor as source in Bianchi type -I and Bianchi VI_0 space-times.

In this paper, we obtain new classes of Bianchi type-I cosmological models with constant deceleration parameters in presence of a conformally invariant scalar field. The plan of the paper is as fallows. We present the metric and field equations in Sect. 2. In Sect. 3, we obtain exact solutions of the field equations by applying the law of variation for Hubble's parameter which yields a constant value of the deceleration parameter. These solutions correspond to singular and non-singular models in two types of cosmologies. The present solutions are new and different than the solutions obtained by Accioly et al.[4] and Shri Ram [8]. The physical and kinematical behaviors of the cosmological models are discussed. Finally, conclusions are summarized in the Section 4. It is shown that the models are compatible with the recent observations in cosmology.

2. THE METRIC AND FIELD EQUATIONS

We consider the diagonal form of the Bianchi type-I metric given by

$$ds^2 = dt^2 - A^2dx^2 + B^2dy^2 + C^2dz^2, \tag{1}$$

where $A(t)$, $B(t)$ and $C(t)$ are the cosmic scale factors. The field equations for the conformally invariant scalar field and the gravitational field from a conformally invariant action integral are (Accioly [4]):

$$R_{\mu\nu}f(u) = g_{\mu\nu}u_{,\alpha}u^{,\alpha} - 4u_{,\mu}u_{,\nu} + 2uu_{;\mu\nu}, \tag{2}$$

$$R = 0, \tag{3}$$

$$u^{;\nu}_{;\mu} = 0 \tag{4}$$

where a comma denotes ordinary derivative and a semicolon denotes covariant derivative and

$$f(u) = 1 - u^2, \tag{5}$$

$$u = (k/6)^{1/2}\phi; \tag{6}$$

ϕ being massless scalar field. Other symbols have their usual meaning.

The field equations (2)-(6) for metric (1), in comoving coordinates, lead to

$$\left(\frac{\ddot{A}}{A} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC}\right) f(u) = \dot{u}^2 + 2u\dot{u}\frac{\dot{A}}{A}, \tag{7}$$

$$\left(\frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC}\right) f(u) = \dot{u}^2 + 2u\dot{u}\frac{\dot{B}}{B}, \tag{8}$$

$$\left(\frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC}\right) f(u) = \dot{u}^2 + 2u\dot{u}\frac{\dot{C}}{C}, \tag{9}$$

$$\left(\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C}\right) f(u) = -3\dot{u}^2 + 2u\ddot{u}, \tag{10}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC} = 0, \tag{11}$$

$$\ddot{u} + \dot{u}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) = 0. \tag{12}$$

Here a dot represent differentiation with respect to time t. The average scale factor R of the metric (1) is defined as

$$R = (ABC)^{1/3}. \tag{13}$$

Equation (12), on integration, gives

$$\dot{u} = \frac{k_1}{R^3} \tag{14}$$

where k_1 is a constant of integration.
The volume scale factor V is given by

$$V = R^3 = ABC. \tag{15}$$

We define the generalized mean Hubble's parameter H as

$$H = \frac{1}{3}(H_1 + H_2 + H_3), \tag{16}$$

where $H_1 = \frac{\dot{A}}{A}$, $H_2 = \frac{\dot{B}}{B}$ and $H_3 = \frac{\dot{C}}{C}$ are directional Hubble's parameters in the direction of x, y and z respectively.
From(13), (15) and (16), we obtain

$$H = \frac{\dot{R}}{R} = \frac{1}{3}(H_1 + H_2 + H_3). \tag{17}$$

For the metric(1) in comoving coordinates the expansion scalar θ, which determines the volume behavior of the fluid, is given by

$$\theta = \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) \tag{18}$$

An important quantity q, the deceleration parameter in cosmology, is given by

$$q = -\frac{R\ddot{R}}{\dot{R}^2}. \tag{19}$$

This parameter measures the rate of expansion at which the expansion of the universe is changing with time in terms of scale factors, and its sign characterizes accelerating or decelerating nature of the universe. In case $q > 0$, the universe decelerates whereas $q < 0$ describes an accelerating universe. It may be noted that though the current observations of SNe Ia and CMB favour accelerating models ($q < 0$), they do not altogether rule out the decelerating ones which are also consistent with these observations (Vishwakarma, [10]).

Now, following the approach of Saha and Rikhvitsky [11], Singh et al.[12] and Shri Ram et al.[13,14], we solve the field equations (7)-(12). Subtracting (7) from (8), (8) from (9) and (7) from (9), we get the following three relations respectively:

$$\frac{B}{A} = d_1 \exp\left[k_1 \int \frac{dt}{(1-u^2)R^3}\right], \tag{20}$$

$$\frac{C}{B} = d_2 \exp\left[k_2 \int \frac{dt}{(1-u^2)R^3}\right], \tag{21}$$

$$\frac{C}{A} = d_3 \exp\left[k_3 \int \frac{dt}{(1-u^2)R^3}\right] \tag{22}$$

where d_1, d_2, d_3 and k_1, k_2, k_3 are constants of integration. From (20)-(22), the metric functions can be written respectively as

$$A = l_1 R \exp\left[\frac{X_1}{3} \int \frac{dt}{(1-u^2)R^3}\right], \tag{23}$$

$$B = l_2 R \exp\left[\frac{X_2}{3} \int \frac{dt}{(1-u^2)R^3}\right], \tag{24}$$

$$C = l_3 R \exp\left[\frac{X_3}{3} \int \frac{dt}{(1-u^2)R^3}\right] \tag{25}$$

where $X_1 = -\frac{(2k_1+k_2)}{3}$, $X_2 = \frac{k_1-k_2}{3}$, $X_3 = \frac{k_1+2k_2}{3}$ and $l_1 = \sqrt[3]{d_1^2 d_2^{-1}}$, $l_2 = \sqrt[3]{d_1 d_2^{-1}}$, $l_3 = \sqrt[3]{d_1 d_2^2}$.
The constants X_1, X_2, X_3 and l_1, l_2, l_3 satisfy the relations

$$X_1 + X_2 + X_3 = 0, \qquad l_1 l_2 l_3 = 1. \tag{26}$$

It is clear, from (23)-(25), that once we get the value of the average scale factor R, we can obtain solutions of the metric functions A, B, C.

3. VARIATION LAW FOR HUBBLE PARAMETER

In order to solve the field equations, let us assume that the Hubble's parameter H is related to the average scale factor R by the relation

$$H = lR^{-n}, \tag{27}$$

where $l > 0$ and $n(\geq)$ are constants. This type of relation has already been discussed by Berman [15], Berman and Gomide [16] for solving FRW models. Such relation gives a constant value of deceleration parameter. Later on, several authors such as Saha [11,17], Singh and Baghel[18], Pradhan and Jotania [19], Pradhan et al. [20,21] and references therein have studied flat FRW and Bianchi type models by using the special law for Hubble parameter that yields constant value of deceleration parameter.

From (17) and (27), we get

$$\dot{R} = lR^{-n+1}, \tag{28}$$

$$\ddot{R} = -l^2(n-1)R^{-2n+1}. \tag{29}$$

From (19), (28) and (29), we obtain

$$q = n - 1. \tag{30}$$

Equation (30) indicates that the deceleration parameter q is constant. Thus the special law of variation of Hubble's parameter (27) yields a constant value of q. The sine of q indicates whether the model inflates or not. The positive sign of q corresponds to standard decelerating model whereas the negative sine indicates inflation. Integrating (28), we obtain the laws of variation for the average scale factor R as

$$R = (nlt + c_1)^{1/n}, \qquad n \neq 0, \tag{31}$$

$$R = c_2 \exp{(lt)}, \tag{32}$$

where c_1 and c_2 are integration constants. Equation (30) implies that the condition for expanding universe is $n = 1 + q > 0$.

4. EXACT SOLUTIONS

We now find the solution of integrals in (23)-(25) in two types of cosmologies for $n \neq 0$ and $n = 0$ explicitly .

4.1 POWER LAW SOLUTION $(n \neq 0)$
From (14) and (31), we obtain

$$u = \frac{k_1}{(n-3)l}\left[(nlt+c_1)^{n-3/n} + k_2\right] \tag{33}$$

where k_2 is an integration constant. Using the values of u and R into (23)-(25), we obtain solutions of A, B and C as follows:

$$A(t) = l_1(nlt+c_1)^{1/n}\left[\frac{(n-3)l + k_1(nlt+c_1)^{n-3/n} + k_2}{(n-3)l - k_1(nlt+c_1)^{n-3/n} - k_2}\right]^{\frac{X_1}{6k_1}}, \tag{34}$$

$$B(t) = l_1(nlt+c_1)^{1/n}\left[\frac{(n-3)l + k_1(nlt+c_1)^{n-3/n} + k_2}{(n-3)l - k_1(nlt+c_1)^{n-3/n} - k_2}\right]^{\frac{X_2}{6k_1}}, \tag{35}$$

$$C(t) = l_1(nlt+c_1)^{1/n}\left[\frac{(n-3)l + k_1(nlt+c_1)^{n-3/n} + k_2}{(n-3)l - k_1(nlt+c_1)^{n-3/n} - k_2}\right]^{\frac{X_3}{6k_1}}. \tag{36}$$

The directional Hubble's parameters H_1, H_2, H_3 and Hubble parameter are obtained as

$$H_1 = \frac{l}{nlt+c_1} + \frac{2X_1(n-3)^2l^2(nlt+c_1)^{-3/n}}{3k_1\left[(n-3)^2l^2 - \{k_1(nlt+c_1)+k_2\}^2\right]}, \tag{37}$$

$$H_2 = \frac{l}{nlt+c_1} + \frac{2X_2(n-3)^2l^2(nlt+c_1)^{-3/n}}{3k_1\left[(n-3)^2l^2 - \{k_1(nlt+c_1)+k_2\}^2\right]}, \tag{38}$$

$$H_3 = \frac{l}{nlt+c_1} + \frac{2X_3(n-3)^2l^2(nlt+c_1)^{-3/n}}{3k_1\left[(n-3)^2l^2 - \{k_1(nlt+c_1)+k_2\}^2\right]}, \tag{39}$$

$$H = \frac{3l}{nlt+c_1}. \tag{40}$$

The expansion scalar $\theta = 3H$ is

$$\theta = \frac{3l}{(nlt+c_1)}. \tag{41}$$

It is observed that the spatial volume is zero and the expansion scalar is infinite at $t = t_1$ where $t_1 = \frac{-c_1}{nl}$, which shows that the universe starts evolving with zero volume at $t = t_1$ with an infinite rate of expansion. The directional scale factors vanish at $t = t_1$ and therefore the model has a point singularity at the initial epoch. The universe exhibits the power-law expansion after the big-bang impulse. As t increases the scale factors and the spatial volume increase but the expansion scalar decreases. Thus the rate of expansion slows down as the cosmic evolution progresses.

The positive value of q indicates that the universe is decelerating. The scalar function u, being infinite at $t = t_1$, decreases as time increases and ultimately becomes a constant for large time.

4.2 EXPONENTIAL SOLUTION ($n = 0$)

In this case, we obtain an exponentially expanding non-singular cosmological model describing the accelerating phase of the present-day universe.

From (14) and (32), we obtain

$$u = k_3 - \frac{k_1}{3lc_2} \exp(-3lt) \tag{42}$$

where k_3 is an integration constant. Using the value of u and R into (23)-(25), we obtain

$$A = l_1 \exp(lt) \left[\frac{2c_2 l + k_1(k_3 - \exp(-3lt))}{2c_2 l - k_1(k_3 - \exp(-3lt))}\right]^{\frac{X_1}{6c_2 k_1}}, \tag{43}$$

$$B = l_2 \exp(lt) \left[\frac{2c_2 l + k_1(k_3 - \exp(-3lt))}{2c_2 l - k_1(k_3 - \exp(-3lt))}\right]^{\frac{X_2}{6c_2 k_1}}, \tag{44}$$

$$C = l_3 \exp(lt) \left[\frac{2c_2 l + k_1(k_3 - \exp(-3lt))}{2c_2 l - k_1(k_3 - \exp(-3lt))}\right]^{\frac{X_3}{6c_2 k_1}}. \tag{45}$$

The directional Hubble's parameters H_1, H_2 and H_3 are obtained as

$$H_1 = l + \frac{3X_1 l^2}{9c_2 l^2 - k_1^2(k_3 - \exp -3lt)^2}, \tag{46}$$

$$H_2 = l + \frac{3X_2 l^2}{9c_2 l^2 - k_1^2(k_3 - \exp -3lt)^2}, \tag{47}$$

$$H_3 = l + \frac{3X_3 l^2}{9c_2 l^2 - k_1^2(k_3 - \exp -3lt)^2}, \tag{48}$$

whereas the average generalized Hubble's parameter $H = l$.
The spatial volume, expansion scalar and deceleration parameter are obtained as

$$V = c_2^3 \exp(3lt), \tag{49}$$

$$\theta = 3l, \tag{50}$$

$$q = -1. \tag{51}$$

From (32), it is observed that as $t \to -\infty$, $R \to 0$ which shows that the universe is infinitely old and has exponential inflationary phase.The directional scale factors and spatial volume of the universe increase exponentially with cosmic time, whereas the mean Hubble parameter and the expansion scalar are constant throughout. As $\to \infty$, the scale factors and volume of the universe become infinitely large, which shows that the universe is dominated by vacuum energy which drives the accelerated expansion of the universe. The scalar function u is a decreasing function of time, which tends to zero for large time.

For $q = -1$, we have $\frac{dH}{dt} = 0$, which implies that the greatest value of the Hubble parameter and the fastest rate of expansion of the universe. Therefore this model is consistent with the observations and may find application in the analysis of the late time evolution of the actual universe.

5. CONCLUSION

In this paper we have investigated spatially homogeneous and an anisotropic Bianchi type-I cosmological models of the universe with constant deceleration parameter within the framework of conformally invariant scalar field. We have obtained solutions by applying the law of variation of Hubble parameter in two types of cosmologies, one with power-law expansion and other one with exponential expansion. The cosmological model with power-law expansion has a point singularity at $t = t_1$ The rate of expansion is a decreasing function of time and finally tends to zero as $t \to -\infty$. The scalar function is infinite at the initial singularity and becomes negligible for large time. The scalar factors and the spatial volume become infinitely large as $t \to -\infty$ which would give essentially an infinite universe. The positive value of deceleration parameter shows that the universe is decelerating one.The cosmological model with exponential expansion has no finite singularity. The directional scale factors are time dependent while the average Hubble parameter is constant. The expansion scalar is constant throughout the time of evolution, which shows that the constant expansion of the universe right from the beginning. The scalar function is always decreasing function of time and tends to zero for large time.Since $q = -1$, this model describes the accelerating phase of the present-day universe. The cosmological models are new and in good agreement with recent observations.

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