On the Numerical Solution of Unsteady Squeezing

MHD Flow of a Second Grade Fluid Between Parallel Plates

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Abstract

The unsteady 2-dimensional flow of electrically conducting, incompressible second grade fluid between two parallel infinite plates approaching or receding from each other symmetrically is studied. A similarity transformation is used to

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reduce the system of partial differential equations to a single fifth-order non-linear differential equation. The resulting non-linear boundary value problem is solved numerically. The effects of appropriate dimensionless parameters on the velocity profiles are studied graphically.

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1. Introduction

The area of unsteady flow in channels or pipes and of fluid near an oscillating or rotating plate has been the focus of several theoretical studies. The reason for this rests with the fact that such problems have many applications in the fields of biomechanics and hydrodynamic lubrication. Most investigations of unsteady flow in channels and of fluid flow near an oscillating plate are confined to velocity or pressure fluctuation imposed axially along the channel [1]. Other problems in channel flows exist when the confining walls have a transverse motion. Such types of flow exist in lubrication when there is squeezing flow between two parallel plates. The tackiness of liquid adhesives also reflects squeeze film effects [2]. The squeeze film geometry has been studied extensively since 1947. Other applications in the biomechanics area relate to squeezing flow between parallel plates and the alternation between contraction and expansion of the blood vessels. In addition, polymer extrusion processes are modeled using squeezing flow of viscous fluids [3].

Neglecting inertial effects, isothermal compressible squeeze films were analyzed by Langlois [4] and Salbu [5]. Thorpe [6] presented an explicit solution of the squeeze flow problem taking inertial terms into account. Later, Gupta and Gupta [7] showed that the solution given by [6] fails to satisfy the boundary conditions. Gupta and Gupta solution, however, is restricted only to small Reynolds number. Squeeze film between two plane annuli with fluid inertia effects has been studied by Elkouh [8]. Some numerical solutions of squeezing flow between parallel plates has been conducted by Verma [9] and later by Singh et. al. [10]. In addition, Hamza [11] has considered suction and injection effects on flow between parallel plates reflecting squeezing flow. Further work on 2nd grade fluid flow between parallel plates has been carried out by Rajagopal & Gupta [12] and by Dandapat & Gupta [13]. Both papers consider the problem of flow between rotating parallel plates. Siddiqui et al. [14,15] has analyzed the unsteady squeezing flow of a second grade fluid between circular plates.

In this work the unsteady 2-dimensional flow of electrically conducting, incompressible second grade fluid between parallel infinite plates that are moving symmetrically about the line of axial symmetry and giving rise to the squeezing flow is studied. The unsteady equations of motion are reduced to a single
numerical nonlinear 5th order ordinary differential after employing a similarity transformation. The resulting non-linear boundary value problem is solved numerically. In section 2 and 3 description and mathematical formulation of the problem is presented. Section 4 discussed the method of solution of the problem. The results are discussed in Section 5 and Section 6 concludes finding of the present work.

2. Description of the problem

The rectilinear unsteady 2-dimensional hydromagnetic squeezing flow of electrically conducting, incompressible second grade fluid between two parallel infinite plates approaching or receding from each other symmetrically is studied. The distance between the plates at any time t is 2a(t). The central axis of the channel is taken as the x-axis and y-axis is normal to it. It is assumed that the plates move symmetrically with respect to the central axis of the channel. A uniform magnetic field \( B = \hat{y} B_0 \) along the y-axis is applied. The fluid is assumed to have electrical conductivity \( \sigma \), density \( \rho \) and kinematics viscosity \( \nu \). The magnetic Reynolds number is assumed to be very small so that the induced magnetic field is negligible [16]. This assumption is reasonable for the liquid metals, e.g. mercury or liquid sodium which are electrically conducting under laboratory conditions. The flow geometry of the problem is shown in Fig. 1.

3. Formulation of the problem

The basic equations governing the motion of a homogeneous incompressible second grade fluid neglecting the thermal effects, are
\( \nabla \cdot \mathbf{V} = 0, \quad (1) \)
\[
\rho \left( \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} \right) = \rho \mathbf{f} + \nabla \cdot \mathbf{T} + \sigma (\mathbf{V} \times \mathbf{B}) \times \mathbf{B}, \quad (2)
\]
where
\[
\mathbf{T} = -p \mathbf{I} + \mu \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_2^2, \quad (3)
\]
and \( \mathbf{V} \) is the velocity vector, \( \mathbf{f} \) is the body force, \( p \) is the pressure, \( \mathbf{T} \) is the stress tensor, \( \mu \) is the coefficient of viscosity and \( \alpha_1 \) and \( \alpha_2 \) are material constants.

The Rivlin-Ericksen tensors \( \mathbf{A}_i \) \( (i = 1, 2) \) are defined as
\[
\mathbf{A}_1 = \nabla \mathbf{V} + (\nabla \mathbf{V})^T, \quad (4)
\]
\[
\mathbf{A}_2 = \frac{d}{dt} \mathbf{A}_1 + \mathbf{A}_1 (\nabla \mathbf{V}) + (\nabla \mathbf{V})^T \mathbf{A}_1, \quad (5)
\]

Now we formulate the equations of motion for unsteady two-dimensional flow.

Assuming that
\[
\mathbf{V} = \left[ u(x,y,t), v(x,y,t), 0 \right], \quad (6)
\]
and introducing the vorticity function \( \omega(x,y,t) \) and generalized pressure gradient \( h(x,y,t) \) as
\[
\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \quad (7)
\]
\[
h = \frac{\rho}{2} \left( u^2 + v^2 \right) + p - \alpha_1 \left[ \mu \nabla^2 u + v \nabla^2 v \right] + \frac{1}{4} (3\alpha_1 + 2\alpha_2) \mathbf{A}_1^2, \quad (8)
\]
where
\[
\mathbf{A}_1^2 = \left[ 4 \left( \frac{\partial u}{\partial x} \right)^2 + 4 \left( \frac{\partial v}{\partial y} \right)^2 + 2 \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right]. \quad (9)
\]

We find that the equations of motion (1) and (2) reduce to
\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (10)
\]
\[
\frac{\partial h}{\partial x} + \rho \left( \frac{\partial u}{\partial t} - v \omega \right) = \left( \mu + \alpha_1 \frac{\partial}{\partial t} \right) \frac{\partial \omega}{\partial y} - \alpha_1 v \nabla^2 \omega - \sigma B_0^2 u, \quad (11)
\]
\[
\frac{\partial h}{\partial y} + \rho \left( \frac{\partial v}{\partial t} + u \omega \right) = \left( \mu + \alpha_1 \frac{\partial}{\partial t} \right) \frac{\partial \omega}{\partial x} + \alpha_1 u \nabla^2 \omega. \quad (12)
\]

Eliminating the generalized pressure \( h \) between the Eqs. (11) and (12) to obtain
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\[ \rho \left( \frac{\partial \omega}{\partial t} + \left( u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) \omega \right) = \left( \mu + \alpha_i \frac{\partial}{\partial t} \right) \nabla^2 \omega + \alpha_i \left( u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) \nabla^2 \omega + \sigma B_0^2 \frac{\partial u}{\partial y}. \]  

The boundary conditions on \( u(x,y,t) \) and \( v(x,y,t) \) are

\[ u(x,a,t) = 0, \quad v(x,a,t) = v_w(t), \quad v(x,0,t) = 0, \quad \frac{\partial u(x,y,t)}{\partial y} \bigg|_{y=0} = 0, \]  

where \( v_w(t) = da/dt \) denotes the velocity of the plates. The first two conditions are due to the no-slip at the upper plate and the remaining two follow from the symmetry of the flow at \( y = 0 \).

If the dimensionless variable \( \eta = y/a(t) \) is introduced, the Eqs. (7) and (10) and (13) become

\[ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial \eta} = \omega, \]  

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{a(t) \partial \eta} = 0, \]  

\[ \rho \left( \frac{\partial \omega}{\partial t} + \left( u \frac{\partial}{\partial x} + v \frac{\partial}{a(t) \partial \eta} \right) \omega \right) = \left( \mu + \alpha_i \frac{\partial}{\partial t} \right) \nabla^2 \omega \]  

\[ + \sigma B_0^2 \frac{\partial u}{a(t) \partial \eta} + \alpha \left( u \frac{\partial}{\partial x} + v \frac{\partial}{a(t) \partial \eta} \right) \nabla^2 \omega. \]  

The boundary conditions on \( u(x,\eta,t) \) and \( v(x,\eta,t) \) are

\[ u(x,1,t) = 0, \quad v(x,1,t) = v_w(t), \quad v(x,0,t) = 0, \quad \frac{\partial u(x,\eta,t)}{\partial \eta} \bigg|_{\eta=0} = 0, \]  

If we define the velocity components as Singh et al. [10] and Birkhoff [17], we define the velocity components as

\[ u = \frac{C-x}{a(t)} v_w(t) f'(\eta), \quad v = v_w(t) f(\eta) \]  

where \( C \) is a constant related to the inlet condition of the channel. The flow is clearly symmetric about \( y = 0 \). It follows that Eq. (15) takes the form

\[ \omega = -\frac{C-x}{a^2(t)} v_w(t) f''(\eta). \]  

Substituting Eqs. (18), (19) in Eqs. (16) and (17), we find that the continuity equation is identically satisfied and Eq. (17) becomes
\[
\frac{av_w}{v} (2f'' - ff'' + \eta f'^{''} - \eta f''^{'}) - \frac{a(dv_w/dt)}{\nu v_w} f'' = -f'' + f''M^2 \\
+ \frac{\alpha_v v_w}{\mu a} \left( \eta f'' + 4 f'' + f''f'' - \nu v_w f'' - \frac{a(dv_w/dt)}{\nu v_w^2} f'' \right),
\]

(21)

where \( M^2 = \sigma B_0^2 / \nu \) and prime denotes differentiation with respect to \( \eta \). The boundary conditions are determined from Eqs. (18) and (19) and are given as

\[
f(1) = 1, \quad f'(1) = 0, \quad f(0) = 0, \quad f''(0) = 0.
\]

(22)

Thus for a similarity solution we define

\[
\frac{av_w}{v} = \text{Re}, \quad \frac{a^2 dv_w / dt}{v v_w} = \text{Re} Q, \quad \frac{\alpha_v v_w}{\mu a} = We,
\]

(23)

where \( \text{Re} \), \( Q \), and \( We \) are functions of \( t \) but for a similarity solution these parameters are considered to be constants. After integrating the first equation of (23), we have

\[
a(t) = (Kt + a_0)^{1/2},
\]

(24)

where \( K \) and \( a_0 \) are constants. When \( K > 0 \) and \( a_0 > 0 \), the plates move apart symmetrically with respect to \( \eta = 0 \). In addition, when \( K < 0 \) and \( a_0 > 0 \), the plates approach each other and squeezing flow exists with similar velocity profile as long as \( (Kt + a_0) > 0 \). From Eqs. (23) and (24), it follows that \( Q = -1 \), and Eq. (21) becomes

\[
f'' + k^2 f'' = l \left( (f - \eta)f''^{''} - ff''^{'}ight) + m \left( (\eta - f)f'' + ff''^{''} \right),
\]

(25)

where

\[
k = \sqrt{\frac{3\text{Re} - M^2}{1 - 5\text{We}}}, \quad l = \frac{\text{Re}}{1 - 5\text{We}}, \quad m = \frac{\text{We}}{1 - 5\text{We}}
\]

(26)

Hence we have a fifth order nonlinear boundary value problem (25) subject to four boundary conditions (22) and is solved numerically.

### 4. Numerical procedure

Consider a simplest boundary value problem
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\[ F(u^{(n)}, u^{(n-1)}, \ldots, u^n, u') = 0, u(a) = A \text{ and } u(b) = B. \tag{27} \]

To solve the boundary value problem the derivatives involved in the problem are approximated by finite differences of appropriate order. This converts the given boundary value problem into a linear system of equations involving values of the function \( u \) at \( a, a + h, a + 2h, \ldots, b \). For higher order accuracy, one should choose \( h \) small. However, this increases the number of equations in the system which in turn increases the computational time. Depending upon the size of this resulting system of linear equations, it can either be solved by exact methods or approximate methods.

In the present problems, we cannot use central differences to approximate the fifth-order derivative for the grid points close to the boundaries. Except for these grid points all the derivatives are central differences and are of 2nd-order accuracy. For the grid points close to the boundaries the fifth-order derivative is inclined to the interior grid points, however, it is still of second-order accuracy. The resulting systems of algebraic equations are solved using successive under relaxation scheme. The difference equations are linearized employing a procedure known as lagging the coefficients [18]. The iterative procedure is repeated until convergence is obtained according to the criterion

\[ \max |u^{(n+1)} - u^{(n)}| < \varepsilon \]

where superscript ‘\( n \)’ represents the number of iterations and ‘\( \varepsilon \)’ is the order of accuracy. In the present case ‘\( \varepsilon \)’ is taken as \( 10^{-16} \).

5. Results and discussion

In this work the unsteady 2-dimensional flow of electrically conducting, incompressible second grade fluid between two parallel infinite plates is considered. The squeezing flow is generated by moving the plates symmetrically about the x-axis. An external uniform magnetic field normal to the axis of symmetry is applied. The unsteady equations of motion are reduced to a single nonlinear 5th order ordinary differential by using a similarity transformation. The resulting nonlinear boundary value problem is solved numerically. Fig. 2 presents the effects of Hartmann number \( M \) on the squeezing flow of the second grade fluid. It is noted that an increase in \( M \) reduces the velocity components monotonically due to the magnetic force against the flow.

The effect of the Reynolds number \( Re \) on velocity profiles is depicted in Fig. 3. In these profiles we fixed the non-Newtonian parameter \( M = 1.4, \ We = 0.05 \) and varied \( Re \) as \( Re = 1.0, 1.5, 2.0 \). It is noted that the normal velocity increases as the Reynolds number increases (Fig 3a). It is also observed that at a given time and for a fixed positive value of Reynolds number the normal velocity increases.
monotonically from $\eta = 0$ to $\eta = 1$. Fig 3b presents effect of Reynolds number on the longitudinal velocity. It is observed that this component of velocity increases near the central axis of the channel but decreases near the walls.

The influence of non-Newtonian parameter $We$ on the normal and longitudinal velocity components is depicted in Fig 4. In this figure we varied $We = 0.0, 0.05, 0.1$ fixing $M = 1.4$, $Re = 1.5$. This observation is similar to that of Reynolds number $Re$ mentioned previously.

![Graphs](a) Normal component of velocity  
(b) Longitudinal component of velocity

Fig. 2. Velocity profiles for various values of $M = 1.4, 1.6, 1.8$ fixing $Re = 1.5, We = 0.05$.

![Graphs](a) Normal component of velocity  
(b) Longitudinal component of velocity

Fig. 3. Velocity profiles for various values of $Re = 1.0, 1.5, 2.0$ fixing $We = 0.05$, $Re = 1.4$.
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We=0.0
We=0.05
We=0.1
Re=1.5
M=1.4
(a) Normal component of velocity
(b) Longitudinal component of velocity

Fig. 4. Velocity profiles for various values of $We = 0.0, 0.05, 0.1$ fixing $M = 1.4$, $Re = 1.5$.

6. Concluding remarks

A similarity solution of the problem for the unsteady flow of electrically conducting, incompressible second grade fluid between two parallel infinite plates approaching or receding from each other has been investigated. It is noted that a similarity solution exists only when the distance between the plates varies as $(Kt + a_0)^{1/2}$, and squeezing flow takes place for $K < 0$ and $a_0 > 0$ as for as $(Kt + a_0) > 0$. Approximate solutions for the fluid velocity have been found numerically. The major finding of the present study can be summarized as follows:

- The transverse magnetic field decreases the fluid motion.
- It is noted that Reynolds number and non-Newtonian parameter have similar effect on the normal and longitudinal velocity components.
- It is found that at a given time and for a fixed positive value of Reynolds number the normal velocity increases monotonically from $\eta = 0$ to $\eta = 1$.
- The longitudinal component of velocity increases near the central axis of the channel but decreases near the walls.
- The results for hydrodynamic unsteady squeezing flow between parallel infinite plates can be recovered by taking $M \rightarrow 0$.
- The results of [7,9] are recovered for $M \rightarrow 0$ and $We \rightarrow 0$.
- The obtained results are valid for all values of Reynolds number and non-Newtonian parameter.
References


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