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A Formulation for Minimizing the Electromagnetic Energy Emitted by a Relativistic Accelerated Charged Particle

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Abstract

We estimate the minimum electromagnetic energy radiated by a relativistic accelerated charged particle; in fact, we estimate the infimum of the above energy. Our starting point is the relativistic version of the Larmor formula for the electromagnetic power emitted by an accelerated charge particle. In addition, extrapolation of the aforementioned approach to the ultrarelativistic domain is done.

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1. Introduction

Studying the dynamics of both classical and quantum charged particles is a very interesting task which, at least from the theoretical point of view, presents a number of open questions [1,3,4,6]. In particular, it is well-known that if a charged particle is accelerated, then it emits electromagnetic energy. This fact exhibits some features which are not still sufficiently investigated. The relativistic case presents a great relevance and will be the domain in which we will develop our theoretical formulation. The radiation losses modify the equation of motion of an accelerated charged particle; this fact is certainly important. In this context, we should remark the relevance of studying the radiation from spontaneous transitions between steady states induced by the interaction of the particle in question with the vacuum radiation field. At this point, let us mention issues related to the Landau-Lifshitz equation in a plane wave [4], the quantum theory of a radiating harmonically bound charge [6] as well as the radiation reaction. Laser-induced effects should be also taken into consideration.

The purpose of this article lies in proposing an analytical formulation for estimating the minimal energy radiated by a relativistic accelerated charged particle. It is clear that minimizing the above energy is desirable for practical reasons because the energy in question is lost (see, for example, refs.[2,5]). Furthermore, considerations upon ultrarelativistic particles will be made. We shall start from the relativistic version of the Larmor formula for the electromagnetic power radiated by a classical accelerated charged particle. From the above relativistic approach, we will deduce an approximate formulation which will be the basis upon which we will derive our results concerning the minimization of the energy losses experienced by the particle. In our formulation, we will employ SI units. On the other hand, we wish to note that our single-particle analysis is really fruitful since, in practice, the involved particles are often fermions as, for instance, electrons forming a gas of non-interacting particles. The fact that the above electrons may be assumed as a gas of non-interacting particles permits to elaborate satisfactory one-electron formulations.

2. Theory

We start from the following expression for energy:

$$W = \int_0^\infty P(t)dt \qquad (1)$$

where $P(t)$ denotes instantaneous power. Here we refer to the relativistic version of the Larmor formula for the power emitted by an accelerated charged particle as follows (in watt):

$$P(t) = \frac{q^2 a^2(t)\gamma^4(t)}{6\pi\varepsilon_0 c^3} \qquad (2)$$

where q is the charge of the particle, c is the speed of light in vacuum, $a(t)$ is the magnitude of the particle acceleration, ε_0 is the dielectric permittivity of vacuum, and $\gamma(t)$ is the Lorentz factor namely:

$$\gamma(t)=\frac{1}{\sqrt{1-\frac{v^2(t)}{c^2}}} \qquad (3)$$

where $v(t)$ is the magnitude of the velocity of the particle.

We have that $a(t)=d(\gamma(t)v(t))/dt$ which, combined with expression (3) and after simple calculations (using $\beta(t)=v(t)/c$), leads to $a(t)=\gamma(t)(1+\beta^2(t)\gamma^2(t))\dot{v}(t)$ which, since $\gamma(t)>1$, implies that $a^2(t)>\dot{v}^2(t)$. Considering this inequality and inserting (3) into (2) as well as replacing the resulting expression into (1), one has:

$$W>\frac{q^2}{6\pi\varepsilon_0 c^3}\int_0^\infty \dot{v}^2(t)\left[1-\frac{v^2(t)}{c^2}\right]^{-2}dt \qquad (4)$$

where, as usually, the overdot denotes derivative with respect to time.

Our aim is minimizing the right-hand side of (4) for $\dot{v}(t)\neq 0$. To get this end, in order to minorize the integral on relationship (4), we start from the following evident inequality:

$$\int_0^\infty \frac{\dot{v}^2(t)}{\left[1-\frac{v^2(t)}{c^2}\right]^2}dt>\int_0^T \frac{\dot{v}^2(t)}{\left[1-\frac{v^2(t)}{c^2}\right]^2}dt \qquad (5)$$

where $T<\infty$.

On the other hand, by using the Cauchy-Schwarz inequality, we get:

$$\int_0^T \frac{\dot{v}^2(t)}{\left[1-\frac{v^2(t)}{c^2}\right]^2}dt\geq\frac{1}{T}\left[\int_0^T \frac{\dot{v}(t)}{1-\frac{v^2(t)}{c^2}}dt\right]^2 \qquad (6)$$

By combining (5) and (6), it follows:

$$\int_0^\infty \frac{\dot{v}^2(t)}{\left[1-\frac{v^2(t)}{c^2}\right]^2}dt>\frac{c^4}{T}\left(\int_{v(0)}^{v(T)}\frac{dv}{c^2-v^2}\right)^2 \qquad (7)$$

The right-hand side of (7) is the greatest (inaccessible) lower bound of the left-hand side of (7), i.e., the (inaccessible) infimum of the left-hand side of (7). Notice that, since the inequality sign in (7) is $>$ rather than $\geq$, then we view the aforementioned infimum as inaccessible. So, by taking into account expression (4) and after calculation of the integral on the right-hand side of (7), one obtains:

$$W_{\min} \approx \frac{q^2}{24\pi\varepsilon_0 cT}\left\{\ln\left\{\frac{[c+v(T)][c-v(0)]}{[c-v(T)][c+v(0)]}\right\}\right\}^2 \qquad (8)$$

Given that the infimum in question is inaccessible, it is not exactly the absolute minimum value of the left-hand side of relation (7) but it is roughly the absolute minimum value (note the sign $\approx$ in formula (8)).

For the ultrarelativistic case, we make $v(T) \to c$ in the numerator of the antilogarithm of (8) and $v(0) \to c$ in the denominator so we get:

$$W_{\min} \approx \frac{q^2}{24\pi\varepsilon_0 cT}\left\{\ln\left[\frac{c-v(0)}{c-v(T)}\right]\right\}^2 \qquad (9)$$

Although from the rigorous mathematical standpoint we have found the minimum value of the energy as the infimum of it, for practical reasons now we are interested in determining some smaller value to minimize W more. To get this end, by applying the Cauchy-Schwarz inequality involving the integral on the right-hand side of (7), we have:

$$\left(\int_{v(0)}^{v(T)} \frac{dv}{c^2 - v^2}\right)^2 \geq [v(T) - v(0)]^{-2}\left(\int_{v(0)}^{v(T)} \frac{dv}{\sqrt{c^2 - v^2}}\right)^4 \qquad (10)$$

After calculating the integral on the right-hand side of (10), from the conjunction of formulae (4), (7) and (10), one finds:

$$W > \frac{cq^2}{6\pi\varepsilon_0 T[v(T)-v(0)]^2}\left\{\arcsin\left[\frac{v(T)}{c}\right] - \arcsin\left[\frac{v(0)}{c}\right]\right\}^4 \qquad (11)$$

From inequality (11) and the obvious condition $v(T) - v(0) < c$, it follows:

$$W > \frac{q^2}{6\pi\varepsilon_0 cT}\left\{\arcsin\left[\frac{v(T)}{c}\right] - \arcsin\left[\frac{v(0)}{c}\right]\right\}^4 \qquad (12)$$

For the ultrarelativistic case, we will obtain an inequality from relationship (11) proceeding in two steps. In the first step, looking at expression (11) with $\phi \equiv \arcsin[v(T)/c]$ and $\theta \equiv \arcsin[v(0)/c]$, we have:

$$\lim_{\phi\to\theta} \frac{\phi - \theta}{\sin\phi - \sin\theta} = \frac{1}{2}\lim_{\phi\to\theta} \frac{\phi - \theta}{\cos\left(\frac{\phi+\theta}{2}\right)\sin\left(\frac{\phi-\theta}{2}\right)} \qquad (13)$$

One has that $\sin((\phi-\theta)/2) \to (\phi-\theta)/2$ as $\phi \to \theta$. Hence, by formula (13), the value of the limit in question is $(\cos\phi)^{-1} = (\cos\theta)^{-1}$ so, by relation (11), this limit becomes the Lorentz factor for $v(T) \approx v(0)$. Incorporating this result into formula (11), one finds straightforwardly:

$$W > \frac{cq^2[v(T)-v(0)]^2}{6\pi\varepsilon_0 T(c^2-u^2)^2} \qquad (14)$$

where u is the approximate common value of $v(T) \approx v(0)$.

In the second step, one starts from the equality $c^2-u^2=(c+u)(c-u)$. In the ultrarelativistic case, we have $u \approx c$ so the above equality reduces to $c^2-u^2 \approx 2c(c-u)$ which, inserted into (14), yields:

$$W > \frac{q^2[v(T)-v(0)]^2}{24\pi\varepsilon_0 cT(c-u)^2} \qquad (15)$$

Finally, we have that $v(T)-v(0) \approx c-u$ so inequality (15) reduces to:

$$W > \frac{q^2}{24\pi\varepsilon_0 cT} \qquad (16)$$

We may continue the minimization process by employing again the Cauchy-Schwarz inequality involving the integral on the right-hand side of (10) but this stage as well as the subsequent stages (relative, of course, to the application of the Cauchy-Schwarz inequality) of the process are irrelevant, the last stage being corresponding to the trivial inequality $W > 0$.

3. Conclusions.- We have performed an estimation process after which formulas (8), (9), (11) and (16) are our main results. As a matter of fact, we have developed an optimizing formulation very valuable from the point of view of mathematical physics. From expression (5), we have used T as finite time interval. In practice, one has that $T >> \tau$, τ being $2/3$ of the duration of the propagation of light through the classical radius of the charged particle. Therefore, it is clear that τ is extremely small (for the electron, we have $\tau \approx 0.63\times10^{-23} s$). More precisely, we have $\tau = q^2/(6\pi\varepsilon_0 m_0 c^3)$ where m_0 stands for the rest-mass of the particle. Finally, we should remark the usefulness (for treating non-interacting particles) of the one-particle approximation done above.

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