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The Dirac Equation According to the Virial Theorem for a Poential $V = kr^n$

Hasan Arslan

Department of Physics, Bingol University, Bingol, Turkey

Abstract

The Dirac equation is written for a chosen potential by using the virial theorem. The derived new form of the Dirac equation does not include a potential term.

Keywords: Virial Theorem, Dirac Equation, Schrodinger Equation

1. Introduction

The application of the virial theorem is done in different fields of science. In this study we apply it to the Dirac equation. The application of the virial theorem to the Dirac equation is done for different purposes as can be search in the references given in this article and in the literature. Previous works are searched. As a result, the virial theorem is applied in a different way to the Dirac equation.The design of the paper is as follows: in section 2, the virial theorem used in quantum mechanics is written; in section 3, the aplication of the virial theorem to the Dirac equation is done; in section 4, the conclusion due the study is given.

2. The Virial Theorem in Quantum Mechanics

The time dependent Schrodinger Equation is given by

$$i\hbar \frac{d\psi}{dt} = H\psi. \tag{1}$$

The derivative of expectation value of an operator A with respect to time is

$$i\hbar \frac{d}{dt}\langle \psi | A | \psi \rangle = \langle \psi | [H, A] | \psi \rangle. \tag{2}$$

Now A is chosen to be $A = \vec{r}.\vec{p}$ [1-7]. Putting this in Eqn. (1), treating A to be time-independent, after some calculations, it is obtained as;

$$\langle \psi | [H, A] | \psi \rangle = 0 \tag{3}$$

Then virial theorem is handled by the calculation

$$[H, A] = \left[\frac{p^2}{2m} + V(r), \vec{r}.\vec{p} \right] = i\hbar\vec{r}.\nabla V - \frac{i\hbar}{m}\vec{p}^2 = \mathrm{i}\hbar\vec{r}.\nabla V - 2i\hbar T = 0 \tag{4}$$

where T is the kinetic energy and V is the potential energy. From here the virial theorem can be written as

$$2\langle T \rangle = \langle \vec{r}.\vec{\nabla} V \rangle \tag{5}$$

This form of the virial theorem with the classical form is used for different purposes in the references [8-24] and the references given therein.
The kinetic energy can be written as

$$T = \frac{p^2}{2m} \tag{6}$$

here p is the momentum operator and m is the mass of the particle under consideration. The momentum operator is

$$p = -i\hbar\nabla \tag{7}$$

Using Eqn. (5) for a potential of the form $V = kr^n$, the kinetic energy is obtained as

$$\langle T \rangle = \frac{n}{2}\langle V(r) \rangle \tag{8}$$

By Eqn (6) and Eqn. (8) the following relation can be written:

$$V(r) = \frac{p^2}{nm} \tag{9}$$

If one is wondered about the classical mechanical virial theorem, he/she is recommended to look the books [25, 26].

3. The Dirac Equation with the Virial Theorem for a Potential Given by $V(r) = kr^n$

The Dirac Equation is

$$i\hbar \frac{\partial \psi}{\partial t} = \left[c\vec{\alpha}.\left(\vec{p} - \frac{e}{c}\vec{A} \right) + \beta mc^2 + e\phi \right]\psi \tag{10}$$

where $\vec{A}$ is a magnetic field,

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix},\ \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{11}$$

and $\vec{\sigma}$ are 2x2 Pauli matrices. For the reader who does not know the Dirac Equation is recommended to see the books [27,28]. The last term in the Eqn. (10) is the potential. This potential term can be inserted by a ptential $V(r) = kr^n$ and by using Eqn. (9). The Eqn. (10) takes the form

$$i\hbar \frac{\partial \psi}{\partial t} = \left[c\vec{\alpha}.\left(\vec{p} - \frac{e}{c}\vec{A} \right) + \beta mc^2 + \frac{p^2}{nm} \right]\psi\ . \tag{12}$$

The wavefunction, in terms of the large and small components, is chosen to be

$$\psi = e^{-\frac{imc^2 t}{\hbar}} \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \tag{13}$$

Using this wave function in Eqn. (12) and writing the small component of the wavefunction in terms of the large component as

$$\chi = \frac{\vec{\sigma}.\vec{\pi}}{2mc}\varphi\ , \tag{14}$$

where $\vec{\pi} = \vec{p} - \frac{e}{c}\vec{A}$, after some calculations two equations are obtained from Eqn. (12) as :

$$i\hbar\frac{\partial\varphi}{\partial t}=c\vec{\sigma}.\vec{\pi}\,\frac{\vec{\sigma}.\vec{\pi}}{2mc}\varphi+\frac{\left(\vec{\pi}+\frac{e}{c}\vec{A}\right)^2}{nm}\varphi\ , \tag{15}$$

$$i\hbar\frac{\vec{\sigma}.\vec{\pi}}{2mc}\frac{\partial\varphi}{\partial t}=c\vec{\sigma}.\vec{\pi}\varphi-2mc^2\frac{\vec{\sigma}.\vec{\pi}}{2mc}\varphi+\frac{\left(\vec{\pi}+\frac{e}{c}\vec{A}\right)^2}{nm}\frac{\vec{\sigma}.\vec{\pi}}{2mc}\varphi\,. \tag{16}$$

Using the identity;

$$\vec{\sigma}.\vec{a}\,\vec{\sigma}.b=\vec{a}.\vec{b}+i\,\vec{\sigma}.axb \tag{17}$$

and vectors cross product;

$$\vec{a}x(\vec{b}x\vec{c})=\vec{b}(\vec{a}.\vec{c})-\vec{c}(\vec{a}.\vec{b}) \tag{18}$$

in Eqn. (15), after some calculations the following equation is obtained:

$$i\hbar\frac{\partial\varphi}{\partial t}=\left[\frac{(n+2)p^2}{2nm}-\frac{e}{2mc}\left(2\vec{S}+L\right)\vec{B}\right]\varphi \tag{19}$$

In calculations that result from Eqn. (15) to Eqn. (19); the $\vec{A}$, $\vec{S}$, and $\vec{L}$ are used as [27]:

$$\vec{A}=\frac{1}{2}\vec{r}x\vec{B},\vec{S}=\frac{1}{2}\hbar\vec{\sigma},\vec{L}=\vec{r}x\vec{p} \tag{20}$$

Also, the field interactions and the higher order terms are neglected in writing Eqn. (19). Eqn. (19), the new form of the Dirac equation, does not include a potential term even if the potential effect is included in it.

4. Conclusion

The Dirac Equation for a stable system is written according to the virial theorem. Although, in the new form of the Dirac equation the potential term cannot be seen, the effect of the potential is represented in the equation. The use of Eqn. (19) will be useful in doing calculations in a stable system in the relativistic situations, and will simplify the calculations.

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