The Frobenius Method on a Second-Order Homogeneous Linear ODEs

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Abstract

In this paper, we show that the one solution of the second-order homogeneous linear differential equation can be obtained by using the method of Frobenius.

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1 Introduction

In mathematics, the Method of Frobenius [2], named for Ferdinand Georg Frobenius, is a method to find an infinite series solution for a second-order ordinary differential equation of the form $x^2y'' + p(x)y' + q(x)y = 0$ with $y' \equiv \frac{dy}{dx}$ and $y'' \equiv \frac{d^2y}{dx^2}$ in the vicinity of the regular singular point $x = 0$. We can divide by x^2 to obtain a differential equation of the form $y'' + \frac{p(x)}{x}y' + \frac{q(x)}{x^2}y = 0$ which will not be solvable with regular power series methods if either $p(x)/x$ or $q(x)/x^2$ are not analytic at $x = 0$. The Frobenius method enables us to make a power series solution to such a differential equation, given that $p(x)$ and $q(x)$ are themselves analytic at 0 or, being analytic elsewhere, both their limits at 0 exist and are finite. Kim [3] used the the method of Frobenius to

find a solution of the Euler-Cauchy equation expressed by differential operator using Laplace transform. Robin [4] derived Frobenius series solution of Fuchs second-Order ordinary differential equations via complex integration. In this paper, we show that the one solution of the second-order homogeneous linear differential equation can be obtained by using the method of Frobenius.

2 The Method of Frobenius

We consider the second-order homogeneous linear differential equation which has variable coefficients. The solution can be obtained by using the method of Frobenius. We show that the power series method can be applied to the process. We derived the one solution of the homogeneous differential equation. We will consider the general solution in the future work.

Definition 2.1. Regular singular point [1]. Given that $x = x_0$ is a singular point of the differential equation $y'' + p(x)y' + q(x)y = 0$. If the functions $(x - x_0)p(x)$ and $(x - x_0)^2q(x)$ are both analytic at x_0 , then x_0 is a regular singular point.

Theorem 2.2. Method of Frobenius [1]. If $x = 0$ is a regular singular point of the second-order homogenous linear differential equation, then the equation has at least one solution of the form

$$\begin{aligned} y(x) &= x^\lambda \sum_{n=0}^{\infty} \xi_n x^n \\ &= \xi_0 x^\lambda + \xi_1 x^{\lambda+1} + \dots + \xi_{n-1} x^{\lambda+n-1} + \xi_n x^{\lambda+n} + \xi_{n+1} x^{\lambda+n+1} + \dots \quad (1) \end{aligned}$$

Where λ and ξ_n are constants in which $n = 0, 1, 2, \dots$. This solution is valid in an interval $0 < x < R$ for some real number R .

By taking the derivative to the above equation, we derived

$$\begin{aligned} y' &= \lambda \xi_0 x^{\lambda-1} + (\lambda + 1) \xi_1 x^\lambda + (\lambda + 2) \xi_2 x^{\lambda+1} + (\lambda + 3) \xi_3 x^{\lambda+2} + \dots \\ &\quad + (\lambda + n) \xi_n x^{\lambda+n-1} + (\lambda + n + 1) \xi_{n+1} x^{\lambda+n} + \dots \quad (2) \end{aligned}$$

The derivative of the above equation becomes

$$\begin{aligned} y'' &= \lambda(\lambda - 1) \xi_0 x^{\lambda-2} + (\lambda + 1) \lambda \xi_1 x^{\lambda-1} + (\lambda + 2)(\lambda + 1) \xi_2 x^{\lambda} + \dots \\ &\quad + (\lambda + n - 1)(\lambda + n - 2) \xi_{n-1} x^{\lambda+n-3} + (\lambda + n)(\lambda + n - 1) \xi_n x^{\lambda+n-2} \\ &\quad + (\lambda + n + 1)(\lambda + n) \xi_{n+1} x^{\lambda+n-1} + \dots \quad (3) \end{aligned}$$

3 The Solution of a Second-Order Homogeneous Linear ODE using Method of Frobenius

Lemma 3.1. *The one solution of the second-order homogeneous linear differential equation $x^2y'' + xy' + (x^2 - \beta)y = 0$ can be expressed by*

$$y(x) = \xi_0 x^{\sqrt{\beta}} \left[1 - \frac{x^2}{\eta_1} + \frac{x^4}{\eta_1 \eta_2} - \frac{x^6}{\eta_1 \eta_2 \eta_3} + \dots \right].$$

where β is a prime number, $\eta_1 = 2^2 + 2(2)\Psi_1 + \Psi_1^2 - \beta$, $\eta_2 = 4^2 + 2(4)\Psi_1 + \Psi_1^2 - \beta$ and $\eta_3 = 6^2 + 2(6)\Psi_1 + \Psi_1^2 - \beta$.

Proof. Since $x = 0$ is a regular singular point, we can apply the method of Frobenius. Substituting Equations (1), (2) and (3) into the left side of the second-order homogeneous linear differential equation $x^2y'' + xy' + (x^2 - \beta)y = 0$. We obtain

$$\begin{aligned} & x^2y'' + xy' + (x^2 - \beta)y \\ &= x^2 [\lambda(\lambda - 1)\xi_0 x^{\lambda-2} + (\lambda + 1)\lambda\xi_1 x^{\lambda-1} + (\lambda + 2)(\lambda + 1)\xi_2 x^{\lambda+1} + \dots \\ &+ (\lambda + n - 1)(\lambda + n - 2)\xi_{n-1} x^{\lambda+n-3} + (\lambda + n)(\lambda + n - 1)\xi_n x^{\lambda+n-2} \\ &+ (\lambda + n + 1)(\lambda + n)\xi_{n+1} x^{\lambda+n-1} + \dots] \\ &+ x [\lambda\xi_0 x^{\lambda-1} + (\lambda + 1)\xi_1 x^{\lambda} + (\lambda + 2)\xi_2 x^{\lambda+1} + (\lambda + 3)\xi_3 x^{\lambda+2} + \dots \\ &+ (\lambda + n - 1)\xi_{n-1} x^{\lambda+n-2} + (\lambda + n)\xi_n x^{\lambda+n-1} + (\lambda + n + 1)\xi_{n+1} x^{\lambda+n} + \dots] \\ &+ (x^2 - \beta) [\xi_0 x^{\lambda} + \xi_1 x^{\lambda+1} + \xi_2 x^{\lambda+2} + \xi_3 x^{\lambda+3} + \dots \\ &+ \xi_{n-1} x^{\lambda+n-1} + \xi_n x^{\lambda+n} + \xi_{n+1} x^{\lambda+n+1} + \dots] = 0. \end{aligned} \quad (4)$$

The above equation can be rearranged by collecting the like power of x and can be explained as

$$\begin{aligned} & x^{\lambda} [(\lambda^2 - \beta)\xi_0] + x^{\lambda+1} [(\lambda + 1)^2 - \beta] \xi_1 \\ &+ x^{\lambda+2} \{[(\lambda + 2)^2 - \beta] \xi_2 + \xi_0\} + \dots \\ &+ x^{\lambda+n} \{[(\lambda + n)^2 - \beta] \xi_n + \xi_{n-2}\} + \dots = 0. \end{aligned} \quad (5)$$

Consequently, we have $(\lambda^2 - \beta)\xi_0 = 0$, $[(\lambda + 1)^2 - \beta] \xi_1 = 0$. And for $n \geq 2$, we obtain

$$\xi_n = -\frac{\xi_{n-2}}{[(\lambda + n)^2 - \beta]}. \quad (6)$$

The indicial equation is $\lambda^2 - \beta = 0$, which has roots $\lambda_1 = \Psi_1$ and $\lambda_2 = \Psi_2$. In this paper, we consider the case for which β is a prime number and because

$\lambda_1 - \lambda_2$ is not an integer. The solution can be derived by substituting $\lambda = \Psi_1$ to the above equations, we obtain $\xi_1 = 0$, and

$$\xi_n = -\frac{\xi_{n-2}}{n^2 + 2\Psi_1 n + \Psi_1^2 - \beta} \text{ for } n \geq 2. \quad (7)$$

Since $\xi_1 = 0$, it follows that $\xi_3 = \xi_5 = \xi_7 = \dots = 0$. Moreover, we can find

$$\xi_2 = -\frac{\xi_0}{2^2 + 2(2)\Psi_1 + \Psi_1^2 - \beta}. \quad (8)$$

$$\xi_4 = \frac{\xi_0}{[2^2 + 2(2)\Psi_1 + \Psi_1^2 - \beta][4^2 + 2(4)\Psi_1 + \Psi_1^2 - \beta]}. \quad (9)$$

And

$$\begin{aligned} \xi_6 = & -\frac{\xi_0}{[2^2 + 2(2)\Psi_1 + \Psi_1^2 - \beta][4^2 + 2(4)\Psi_1 + \Psi_1^2 - \beta]} \\ & \times \frac{1}{[6^2 + 2(6)\Psi_1 + \Psi_1^2 - \beta]}. \end{aligned} \quad (10)$$

In the similar process, we can find ξ_{2k} for $k = 4, 5, \dots$. As a consequence, by substituting equations (8), (9) and (10) to equation (1), we have the one solution of the differential equation as

$$y_1(x) = \xi_0 x^{\Psi_1} \left[1 - \frac{x^2}{\eta_1} + \frac{x^4}{\eta_1 \eta_2} - \frac{x^6}{\eta_1 \eta_2 \eta_3} + \dots \right]. \quad (11)$$

where $\eta_1 = 2^2 + 2(2)\Psi_1 + \Psi_1^2 - \beta$, $\eta_2 = 4^2 + 2(4)\Psi_1 + \Psi_1^2 - \beta$ and $\eta_3 = 6^2 + 2(6)\Psi_1 + \Psi_1^2 - \beta$. The general solution of ODE is of the form $y = y_1(x) + y_2(x)$. For the solution $y_2(x)$, we will leave for the further work. $\square$

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