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Some Identities for the Higher-order q -Bernoulli Polynomials of the Second Kind under the Third Dihedral Group D_3

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Abstract

In this paper, we consider the higher-order q -Bernoulli polynomials of the second kind and investigate some symmetric identities under the third Dihedral group D_3 which are derived from multivariate p -adic invariant integral on $\mathbb{Z}_p$.

Keywords: p -adic invariant integral on $\mathbb{Z}_p$, higher-order q -Bernoulli numbers and polynomials of the second kind

1. INTRODUCTION

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will, respectively, denote the ring of p -adic rational integers, the field of p -adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = 1/p$ and let q be an indeterminate in $\mathbb{C}_p$ with $|1 - q|_p < p^{-1/(p-1)}$. As is well known, the q -number of x is defined by $[x]_q = (1 - q^x)/(1 - q)$. Note that $\lim_{q \rightarrow 1} [x]_q = x$. Let us assume that $f(x)$ is uniformly differentiable function on $\mathbb{Z}_p$. The p -adic invariant integral on $\mathbb{Z}_p$ is defined to be

$$\begin{aligned} \int_{\mathbb{Z}_p} f(x) d\mu_0(x) &= \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} f(x) \mu_0(x + p^N \mathbb{Z}_p) \\ &= \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \quad (\text{see [8, 10, 12]}). \end{aligned} \quad (1)$$

From (1), we have

$$\int_{\mathbb{Z}_p} f(x+1) d\mu_0(x) - \int_{\mathbb{Z}_p} f(x) d\mu_0(x) = f'(0), \quad (2)$$

where $f'(0) = df(x)/dx|_{x=0}$. By (2), we easily get

$$\int_{\mathbb{Z}_p} \exp(t(x+y)) d\mu_0(y) = \frac{t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \quad (3)$$

where $B_n(x)$ are called Bernoulli polynomials. When $x = 0$, $B_n = B_n(0)$ are called the Bernoulli numbers (see [8, 9, 10]).

Recently, several authors have studied variously q -analogue of Bernoulli numbers and polynomials (see [1-20]). In [8], D. S. Kim and T. Kim have considered q -Bernoulli polynomials of the second kind as follows:

$$\int_{\mathbb{Z}_p} \exp([x+y]_q t) d\mu_0(y) = \sum_{n=0}^{\infty} \beta_{n,q}(x) \frac{t^n}{n!}. \quad (4)$$

When $x = 0$, $\beta_{n,q} = \beta_{n,q}(0)$ are called the q -Bernoulli numbers of the second kind. From (4), we can derive the following recurrence relation:

$$\beta_{0,q} = 1, (q\beta_q + 1)^n - \beta_{n,q} = \begin{cases} \frac{\log q}{q-1}, & \text{if } n = 1 \\ 0, & \text{if } n > 1, \end{cases} \quad (5)$$

with the usual convention about replacing β_q^n by $\beta_{n,q}$ (see [6,7,8]).

By (4) and (5), we get

$$\beta_{n,q}(x) = \frac{\log q}{q-1} \frac{1}{(1-q)^n} \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{k}{[k]_q} q^{kx}, \quad (\text{see [8]}). \quad (6)$$

For $r \in \mathbb{N}$, we consider the higher-order q -Bernoulli polynomials of the second kind as follows :

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \exp \left(\left[x + \sum_{\ell=1}^r y_\ell \right]_q \times t \right) d\mu_0(y_1) \cdots d\mu_0(y_r) = \sum_{n=0}^{\infty} \beta_{n,q}^{(r)}(x) \frac{t^n}{n!}. \quad (7)$$

When $x = 0$, $\beta_{n,q}^{(r)} = \beta_{n,q}^{(r)}(0)$ are called the Bernoulli numbers of order r . Thus, by (7), we get

$$\begin{aligned} \beta_{n,q}^{(r)}(x) &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[x + \sum_{\ell=1}^r y_\ell \right]_q^n d\mu_0(y_1) \cdots d\mu_0(y_r) \\ &= \left(\frac{\log q}{q-1} \right)^r \frac{1}{(1-q)^n} \sum_{k=0}^n \binom{n}{k} (-1)^k \left(\frac{k}{[k]_q} \right)^r q^{kx}. \end{aligned} \quad (8)$$

In this paper, we investigate some symmetric identities for the higher-order q -Bernoulli polynomials of the second kind under the third Dihedral group D_3 .

2. SOME IDENTITIES FOR THE HIGHER-ORDER q -BERNOULLI POLYNOMIALS OF THE SECOND KIND

For $w_1, w_2, w_3 \in \mathbb{Z}$ with $w_1 \geq 1, w_2 \geq 1, w_3 \geq 1$, let

$$f_1(y_1, \dots, y_r) = \exp \left\{ \left[w_1 w_2 w_3 x + \sum_{\ell=1}^r (w_2 w_3 y_\ell + w_1 w_3 i_\ell + w_1 w_2 j_\ell) \right]_q \times t \right\}$$

and

$$g_1(y_1, \dots, y_r) = \exp \left\{ \left[w_1 w_2 w_3 x + \sum_{\ell=1}^r (w_2 w_3 (h_\ell + w_1 y_\ell) + w_1 w_3 i_\ell + w_1 w_2 j_\ell) \right]_q \times t \right\}.$$

Then we see that

$$\begin{aligned}
 & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} f_1(y_1, \dots, y_r) d\mu_0(y_1) \cdots d\mu_0(y_r) \\
 &= \lim_{N \rightarrow \infty} \left(\frac{1}{p^N} \right)^r \sum_{y_1, \dots, y_r=0}^{p^r-1} f_1(y_1, \dots, y_r) \\
 &= \lim_{N \rightarrow \infty} \left(\frac{1}{w_1 p^N} \right)^r \sum_{h_1, \dots, h_r=0}^{w_1-1} \sum_{y_1, \dots, y_r=0}^{p^N-1} g_1(y_1, \dots, y_r),
 \end{aligned} \tag{9}$$

From (9), we can derive the following equation:

$$\begin{aligned}
 & \frac{1}{(w_2 w_3)^r} \sum_{i_1, \dots, i_r=0}^{w_2-1} \sum_{j_1, \dots, j_r=0}^{w_3-1} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} f_1(y_1, \dots, y_r) d\mu_0(y_1) \cdots d\mu_0(y_r) \\
 &= \lim_{N \rightarrow \infty} \left(\frac{1}{w_1 w_2 w_3 p^N} \right)^r \sum_{i_1, \dots, i_r=0}^{w_2-1} \sum_{j_1, \dots, j_r=0}^{w_3-1} \sum_{h_1, \dots, h_r=0}^{w_1-1} \sum_{y_1, \dots, y_r=0}^{p^N-1} g_1(y_1, \dots, y_r).
 \end{aligned} \tag{10}$$

By the same method as (10), we get

$$\begin{aligned}
 & \frac{1}{(w_1 w_3)^r} \sum_{i_1, \dots, i_r=0}^{w_3-1} \sum_{j_1, \dots, j_r=0}^{w_1-1} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} f_2(y_1, \dots, y_r) d\mu_0(y_1) \cdots d\mu_0(y_r) \\
 &= \lim_{N \rightarrow \infty} \left(\frac{1}{w_1 w_2 w_3 p^N} \right)^r \sum_{i_1, \dots, i_r=0}^{w_3-1} \sum_{j_1, \dots, j_r=0}^{w_1-1} \sum_{h_1, \dots, h_r=0}^{w_2-1} \sum_{y_1, \dots, y_r=0}^{p^N-1} g_2(y_1, \dots, y_r).
 \end{aligned} \tag{11}$$

where

$$f_2(y_1, \dots, y_r) = \exp \left\{ \left[w_1 w_2 w_3 x + \sum_{\ell=1}^r (w_3 w_1 y_\ell + w_2 w_1 i_\ell + w_2 w_3 j_\ell) \right]_q \times t \right\}$$

and

$$g_2(y_1, \dots, y_r) = \exp \left\{ \left[w_1 w_2 w_3 x + \sum_{\ell=1}^r (w_3 w_1 (h_\ell + w_2 y_\ell) + w_2 w_1 i_\ell + w_2 w_3 j_\ell) \right]_q \times t \right\}.$$

Therefore, by (10) and (11), we obtain the following theorem.

Theorem 2.1. *Let $w_1, w_2, w_3 \in \mathbb{Z}$ with $w_1 \geq 1, w_2 \geq 1, w_3 \geq 1$. Then the following expressions*

$$\frac{1}{(w_{\sigma(2)} w_{\sigma(3)})^r} \sum_{i_1, \dots, i_r=0}^{w_{\sigma(2)}-1} \sum_{j_1, \dots, j_r=0}^{w_{\sigma(3)}-1} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} f(y_1, \dots, y_r) d\mu_0(y_1) \cdots d\mu_0(y_r)$$

are the same for any $\sigma \in D_3$, where

$$f(y_1, \dots, y_r) = \exp \left\{ \left[w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)} x + \sum_{\ell=1}^r (w_{\sigma(2)} w_{\sigma(3)} y_{\ell} + w_{\sigma(1)} w_{\sigma(3)} i_{\ell} + w_{\sigma(1)} w_{\sigma(2)} j_{\ell}) \right]_q \times t \right\}.$$

By (8), we get

$$\begin{aligned} & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \exp \left\{ \left[w_1 w_2 w_3 x + \sum_{\ell=1}^r (w_2 w_3 y_{\ell} + w_1 w_3 i_{\ell} + w_1 w_2 j_{\ell}) \right]_q \times t \right\} d\mu_0(y_1) \cdots d\mu_0(y_r) \\ &= \sum_{n=0}^{\infty} [w_2 w_3]_q^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[w_1 x + \sum_{\ell=1}^r \left(y_{\ell} + \frac{w_1}{w_2} i_{\ell} + \frac{w_1}{w_3} j_{\ell} \right) \right]_{q^{w_2 w_3}}^n d\mu_0(y_1) \cdots d\mu_0(y_r) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} [w_2 w_3]_q^n \beta_{n, q^{w_2 w_3}}^{(r)} \left(w_1 x + \sum_{\ell=1}^r \left(\frac{w_1}{w_2} i_{\ell} + \frac{w_1}{w_3} j_{\ell} \right) \right). \end{aligned} \quad (12)$$

Therefore, by Theorem 1 and (12), we obtain the following theorem.

Theorem 2.2. Let $w_1, w_2, w_3 \in \mathbb{Z}$ with $w_1 \geq 1, w_2 \geq 1, w_3 \geq 1$. For any non-negative integer n , the following expressions

$$\frac{[w_{\sigma(2)} w_{\sigma(3)}]_q^n}{(w_{\sigma(2)} w_{\sigma(3)})^r} \sum_{i_1, \dots, i_r=0}^{w_{\sigma(2)}-1} \sum_{i_1, \dots, i_r=0}^{w_{\sigma(3)}-1} \beta_{n, q^{w_{\sigma(2)} w_{\sigma(3)}}}^{(r)} \left(w_{\sigma(1)} x + \sum_{\ell=1}^r \left(\frac{w_{\sigma(1)}}{w_{\sigma(2)}} i_{\ell} + \frac{w_{\sigma(1)}}{w_{\sigma(3)}} j_{\ell} \right) \right)$$

are the same for any $\sigma \in D_3$.

Now, we observe that

$$\begin{aligned} & \left[w_1 x + \sum_{\ell=1}^r \left(y_{\ell} + \frac{w_1}{w_2} i_{\ell} + \frac{w_1}{w_3} j_{\ell} \right) \right]_{q^{w_2 w_3}} \\ &= \frac{1 - q^{\sum_{\ell=1}^r (w_1 w_3 i_{\ell} + w_1 w_2 j_{\ell})}}{1 - q^{w_2 w_3}} + q^{\sum_{\ell=1}^r (w_1 w_3 i_{\ell} + w_1 w_2 j_{\ell})} \left[\sum_{\ell=1}^r y_{\ell} + w_1 x \right]_{q^{w_2 w_3}} \\ &= \frac{[w_1]_q}{[w_2 w_3]_q} \left[\sum_{\ell=1}^r (w_3 i_{\ell} + w_2 j_{\ell}) \right]_{q^{w_1}} + q^{\sum_{\ell=1}^r (w_1 w_3 i_{\ell} + w_1 w_2 j_{\ell})} \left[\sum_{\ell=1}^r y_{\ell} + w_1 x \right]_{q^{w_2 w_3}}. \end{aligned} \quad (13)$$

From (13), we can derive the following equation :

$$\begin{aligned}
 & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[w_1 x + \sum_{\ell=1}^r \left(y_\ell + \frac{w_1}{w_2} i_\ell + \frac{w_1}{w_3} j_\ell \right) \right]_{q^{w_2 w_3}}^n d\mu_0(y_1) \cdots d\mu_0(y_r) \\
 &= \sum_{k=0}^n \binom{n}{k} \left(\frac{[w_1]_q}{[w_2 w_3]_q} \right)^{n-k} \left[\sum_{\ell=1}^r (w_3 i_\ell + w_2 j_\ell) \right]_{q^{w_1}}^{n-k} q^{k \sum_{\ell=1}^r (w_1 w_3 i_\ell + w_1 w_2 j_\ell)} \\
 & \quad \times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[\sum_{\ell=1}^r y_\ell + w_1 x \right]_{q^{w_2 w_3}}^k d\mu_0(y_1) \cdots d\mu_0(y_r) \\
 &= \sum_{k=0}^n \binom{n}{k} \left(\frac{[w_1]_q}{[w_2 w_3]_q} \right)^{n-k} \left[\sum_{\ell=1}^r (w_3 i_\ell + w_2 j_\ell) \right]_{q^{w_1}}^{n-k} \\
 & \quad \times q^{k \sum_{\ell=1}^r (w_1 w_3 i_\ell + w_1 w_2 j_\ell)} \beta_{k, q^{w_2 w_3}}^{(r)}(w_1 x)
 \end{aligned} \tag{14}$$

By (10), (12) and (14), we get

$$\begin{aligned}
 & \frac{[w_2 w_3]_q^n}{(w_2 w_3)^r} \sum_{i_1, \dots, i_r=0}^{w_2-1} \sum_{j_1, \dots, j_r=0}^{w_3-1} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \\
 & \quad \times \left[w_1 x + \sum_{\ell=1}^r \left(y_\ell + \frac{w_1}{w_2} i_\ell + \frac{w_1}{w_3} j_\ell \right) \right]_{q^{w_2 w_3}}^n d\mu_0(y_1) \cdots d\mu_0(y_r) \\
 &= \sum_{k=0}^n \binom{n}{k} \frac{[w_2 w_3]_q^k}{(w_2 w_3)^r} [w_1]_q^{n-k} \beta_{k, q^{w_2 w_3}}^{(r)}(w_1 x) \sum_{i_1, \dots, i_r=0}^{w_2-1} \sum_{j_1, \dots, j_r=0}^{w_3-1} \\
 & \quad \times q^{k \sum_{\ell=1}^r (w_1 w_3 i_\ell + w_1 w_2 j_\ell)} \left[\sum_{\ell=1}^r (w_3 i_\ell + w_2 j_\ell) \right]_{q^{w_1}}^{n-k} \\
 &= \sum_{k=0}^n \binom{n}{k} \frac{[w_2 w_3]_q^k}{(w_2 w_3)^r} [w_1]_q^{n-k} \beta_{k, q^{w_2 w_3}}^{(r)}(w_1 x) T_{n, q^{w_1}}^{(r)}(w_2, w_3 | k),
 \end{aligned} \tag{15}$$

where

$$T_{n, q}^{(r)}(w_1, w_2 | k) = \sum_{i_1, \dots, i_r=0}^{w_1-1} \sum_{j_1, \dots, j_r=0}^{w_2-1} q^{k \sum_{\ell=1}^r (w_2 i_\ell + w_1 j_\ell)} \left[\sum_{\ell=1}^r (w_2 i_\ell + w_1 j_\ell) \right]_q^{n-k}.$$

As this expression is an invariant under third Dihedral group D_3 , we have the following theorem.

Theorem 2.3. *Let w_1, w_2, w_3 be any positive integers. Then, for any non-negative integer n , the following expressions*

$$\sum_{k=0}^n \binom{n}{k} \frac{[w_{\sigma(2)}w_{\sigma(3)}]_q^k}{(w_{\sigma(2)}w_{\sigma(3)})^r} [w_{\sigma(1)}]_q^{n-k} \beta_{k,q}^{(r)w_{\sigma(2)}w_{\sigma(3)}}(w_{\sigma(1)}x) T_{n,q}^{(r)w_{\sigma(1)}}(w_{\sigma(2)}, w_{\sigma(3)}|k)$$

are all the same for any $\sigma \in D_3$.

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