Symmetric Properties for the Generalized (h, q) -Tangent Polynomials

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Abstract

In this paper, we study the symmetry for the generalized (h, q) -tangent numbers $T_{n,\chi,q}^{(h)}$ and polynomials $T_{n,\chi,q}^{(h)}(x)$. We give some interesting identities of the power sums and the generalized polynomials $T_{n,\chi,q}^{(h)}(x)$ using the symmetric properties for the p -adic invariant integral on $\mathbb{Z}_p$.

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1 Introduction

Euler numbers, Euler polynomials, Bernoulli numbers, Bernoulli polynomials, tangent numbers, and tangent polynomials possess many interesting properties and arise in many areas of mathematics and physics(see [1-6]). Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p -adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p -adic rational numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one

talks of q -extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p -adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q - 1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $g \in UD(\mathbb{Z}_p)$ the fermionic p -adic invariant integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-1}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N \rightarrow \infty} \sum_{0 \leq x < p^N} g(x) (-1)^x, \quad (\text{see [1]}). \quad (1.1)$$

If we take $g_n(x) = g(x + n)$ in (1.1), then we see that

$$I_{-1}(g_n) = (-1)^n I_{-1}(g) + 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} g(l). \quad (1.2)$$

Let a fixed positive integer d with $(p, d) = 1$, set

$$X = X_d = \varprojlim_N (\mathbb{Z}/dp^N \mathbb{Z}), \quad X_1 = \mathbb{Z}_p, \quad X^* = \bigcup_{\substack{0 < a < dp \\ (a, p) = 1}} a + dp\mathbb{Z}_p,$$

$$a + dp^N \mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^N$. It is easy to see that

$$I_{-1}(g) = \int_X g(x) d\mu_{-1}(x) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x). \quad (1.3)$$

Let χ be Dirichlet's character with conductor $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$. In [5], we introduced the generalized (h, q) -tangent numbers $T_{n, \chi, q}^{(h)}$ and polynomials $T_{n, \chi, q}^{(h)}(x)$ attached to χ . The generalized (h, q) -tangent numbers $T_{n, \chi, q}^{(h)}$ attached to χ are defined by the generating function:

$$\frac{2 \sum_{a=0}^{d-1} \chi(a) (-1)^a q^{ha} e^{2at}}{q^{hd} e^{2dt} + 1} = \sum_{n=0}^{\infty} T_{n, \chi, q}^{(h)} \frac{t^n}{n!}, \quad \text{cf. [5]}. \quad (1.4)$$

We consider the generalized (h, q) -tangent polynomials $T_{n, \chi, q, w}^{(h)}(x)$ attached to χ as follows:

$$\left(\frac{2 \sum_{a=0}^{d-1} \chi(a) (-1)^a q^{ha} e^{2at}}{q^{hd} e^{2dt} + 1} \right) e^{xt} = \sum_{n=0}^{\infty} T_{n, \chi, q}^{(h)}(x) \frac{t^n}{n!}. \quad (1.5)$$

Theorem 1.1 ([5]) *For positive integers n and $h \in \mathbb{Z}$, we have*

$$T_{n, \chi, q}^{(h)}(x) = \int_X \chi(y) q^{hy} (2y + x)^n d\mu_{-1}(y).$$

Corollary 1.2 ([5]) For positive integers n and $h \in \mathbb{Z}$, we have

$$T_{n,\chi,q}^{(h)} = \int_X \chi(y) q^{hy} (2y)^n d\mu_{-1}(y).$$

Theorem 1.3 ([5]) For positive integers n and $h \in \mathbb{Z}$, we have

$$T_{n,\chi,q}^{(h)}(x) = \sum_{l=0}^n \binom{n}{l} T_{l,\chi,q}^{(h)}.$$

2 Symmetry for for the generalized (h, q) -tangent polynomials

In this section, we assume that $q \in \mathbb{C}_p$. We obtain some interesting identities of the power sums and the generalized polynomials $T_{n,\chi,q}^{(h)}(x)$ using the symmetric properties for the p -adic invariant integral on $\mathbb{Z}_p$. If n is odd from (1.2), we have

$$I_{-1}(g_n) + I_{-1}(g) = 2 \sum_{k=0}^{n-1} (-1)^k g(k) \quad (\text{see [1], [2], [3], [5]}). \quad (2.1)$$

It will be more convenient to write (3.1) as the equivalent integral form

$$\int_{\mathbb{Z}_p} g(x+n) d\mu_{-1}(x) + \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = 2 \sum_{k=0}^{n-1} (-1)^k g(k). \quad (2.2)$$

Substituting $g(x) = \chi(x) q^{hx} e^{2xt}$ into the above, we obtain

$$\begin{aligned} & \int_X \chi(x+n) q^{h(x+n)} e^{(2x+2n)t} d\mu_{-1}(x) + \int_X \chi(x) q^{hx} e^{2xt} d\mu_{-1}(x) \\ &= 2 \sum_{j=0}^{n-1} (-1)^j \chi(j) q^{hj} e^{2jt}. \end{aligned} \quad (2.3)$$

For $k \in \mathbb{N} \cup \{0\}$, let us define the power sums $\mathcal{T}_{k,\chi,q}^{(h)}(n)$ as follows:

$$\mathcal{T}_{k,\chi,q}^{(h)}(n) = \sum_{l=0}^n (-1)^l \chi(l) q^{hl} (2l)^k. \quad (2.4)$$

After some elementary calculations, we have

$$\begin{aligned} & \sum_{m=0}^{\infty} \left(\int_X \chi(x) q^{h(x+nd)} (2x+2nd)^m d\mu_{-1}(x) + \int_X \chi(x) q^{hx} (2x)^m d\mu_{-1}(x) \right) \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty} \left(2 \sum_{j=0}^{nd-1} (-1)^j \chi(j) q^{hj} (2j)^m \right) \frac{t^m}{m!} \end{aligned} \quad (2.5)$$

From the above, we get

$$\begin{aligned} & \int_X \chi(x) \zeta^{x+nd} e^{(2x+2nd)t} d\mu_{-1}(x) + \int_X \chi(x) \zeta^x e^{2xt} d\mu_{-1}(x) \\ &= \frac{2 \int_X \chi(x) \zeta^x e^{2xt} d\mu_{-1}(x)}{\int_X \zeta^{ndx} e^{2ndtx} d\mu_{-1}(x)}. \end{aligned} \quad (2.6)$$

By using (2.5) and (2.6), we arrive at the following theorem:

Theorem 2.1 *Let n be odd positive integer. Then we obtain*

$$\frac{2 \int_X \chi(x) q^{hx} e^{2xt} d\mu_{-1}(x)}{\int_X q^{hndx} e^{2ndtx} d\mu_{-1}(x)} = \sum_{m=0}^{\infty} (2\mathcal{T}_{m,\chi,q}^{(h)}(nd-1)) \frac{t^m}{m!}.$$

Let w_1 and w_2 be odd positive integers. Then we set

$$\begin{aligned} S(w_1, w_2) = & \frac{\int_X \int_X \chi(x_1) \chi(x_2) q^{h(w_1x_1+w_2x_2)} e^{(2w_1x_1+2w_2x_2+w_1w_2x)t} d\mu_{-1}(x_1) d\mu_{-1}(x_2)}{\int_X q^{hw_1w_2dx} e^{2w_1w_2dxt} d\mu_{-1}(x)}. \end{aligned} \quad (2.7)$$

By Theorem 2.1 and (2.7), after elementary calculations, we obtain

$$S(w_1, w_2) = \left(\frac{1}{2} \sum_{m=0}^{\infty} T_{m,\chi,q^{w_1}}^{(h)}(w_2x) w_1^m \frac{t^m}{m!} \right) \left(2 \sum_{m=0}^{\infty} \mathcal{T}_{m,\chi,q^{w_2}}^{(h)}(w_1d-1) w_2^m \frac{t^m}{m!} \right).$$

By using Cauchy product in the above, we have

$$S(w_1, w_2) = \sum_{m=0}^{\infty} \left(\sum_{j=0}^m \binom{m}{j} T_{j,\chi,q^{w_1}}^{(h)}(w_2x) w_1^j \mathcal{T}_{m-j,\chi,q^{w_2}}^{(h)}(w_1d-1) w_2^{m-j} \right) \frac{t^m}{m!}. \quad (2.8)$$

From the symmetry of $S(w_1, w_2)$ in w_1 and w_2 , we also see that

$$S(w_1, w_2) = \left(\frac{1}{2} \sum_{m=0}^{\infty} T_{m,\chi,q^{w_2}}^{(h)}(w_1x) w_2^m \frac{t^m}{m!} \right) \left(2 \sum_{m=0}^{\infty} \mathcal{T}_{m,\chi,q^{w_1}}^{(h)}(w_2d-1) w_1^m \frac{t^m}{m!} \right).$$

Thus we obtain

$$S(w_1, w_2) = \sum_{m=0}^{\infty} \left(\sum_{j=0}^m \binom{m}{j} T_{j,\chi,q^{w_2}}^{(h)}(w_1x) w_2^j \mathcal{T}_{m-j,\chi,q^{w_1}}^{(h)}(w_2d-1) w_1^{m-j} \right) \frac{t^m}{m!} \quad (2.9)$$

By comparing coefficients $\frac{t^m}{m!}$ in the both sides of (2.8) and (2.9), we arrive at the following theorem:

Theorem 2.2 *Let w_1 and w_2 be odd positive integers. Then we obtain*

$$\begin{aligned} & \sum_{j=0}^m \binom{m}{j} w_1^{m-j} w_2^j T_{j, \chi, q^{w_2}}^{(h)}(w_1 x) \mathcal{T}_{m-j, \chi, q^{w_1}}^{(h)}(w_2 d - 1) \\ &= \sum_{j=0}^m \binom{m}{j} w_1^j w_2^{m-j} T_{j, \chi, q^{w_1}}^{(h)}(w_2 x) \mathcal{T}_{m-j, \chi, q^{w_2}}^{(h)}(w_1 d - 1), \end{aligned}$$

where $T_{k, \chi, q}^{(h)}(x)$ and $\mathcal{T}_{m, \chi, q}^{(h)}(k)$ denote the generalized (h, q) -tangent polynomials and the alternating sums of powers of consecutive (h, q) -integers, respectively.

By Theorem 2.2 and Theorem 1.3, we have the following corollary.

Corollary 2.3 *Let w_1 and w_2 be odd positive integers. Then we obtain*

$$\begin{aligned} & \sum_{j=0}^m \sum_{k=0}^j \binom{m}{j} \binom{j}{k} w_1^{m-k} w_2^j x^{j-k} T_{k, \chi, q^{w_2}}^{(h)} \mathcal{T}_{m-j, \chi, q^{w_1}}^{(h)}(w_2 d - 1) \\ &= \sum_{j=0}^m \sum_{k=0}^j \binom{m}{j} \binom{j}{k} w_1^j w_2^{m-k} x^{j-k} T_{k, \chi, q^{w_1}}^{(h)} \mathcal{T}_{m-j, \chi, q^{w_2}}^{(h)}(w_1 d - 1). \end{aligned}$$

Now we will derive another interesting identities for the generalized (h, q) -tangent polynomials using the symmetric property of $S(w_1, w_2)$. By (2.7), after elementary calculations, we obtain

$$\begin{aligned} & S(w_1, w_2) \\ &= \left(\frac{1}{2} e^{w_1 w_2 x t} \int_X \chi(x_1) q^{h w_1 x_1} e^{2 x_1 w_1 t} d\mu_{-1}(x_1) \right) \left(2 \sum_{j=0}^{w_1 d - 1} (-1)^j \chi(j) q^{w_2 h j} e^{2 j w_2 t} \right) \\ &= \sum_{j=0}^{w_1 d - 1} (-1)^j \chi(j) q^{w_2 h j} \int_X \chi(x_1) q^{h w_1 x_1} e^{\left(2 x_1 + w_2 x + \frac{2 j w_2}{w_1} \right) (w_1 t)} d\mu_{-1}(x_1) \\ &= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{w_1 d - 1} (-1)^j \chi(j) q^{w_2 h j} T_{n, \chi, q^{w_1}}^{(h)} \left(w_2 x + \frac{2 j w_2}{w_1} \right) w_1^n \right) \frac{t^n}{n!}. \end{aligned} \tag{2.10}$$

By using the symmetry property in (2.10), we also have

$$\begin{aligned}
 S(w_1, w_2) &= \left(\frac{1}{2} e^{w_1 w_2 x t} \int_X \chi(x_2) q^{h w_2 x_2} e^{2 x_2 w_2 t} d\mu_{-1}(x_2) \right) \left(2 \sum_{j=0}^{w_2 d-1} (-1)^j \chi(j) q^{w_1 h j} e^{2 j w_1 t} \right) \\
 &= \sum_{j=0}^{w_2 d-1} (-1)^j \chi(j) q^{w_1 h j} \int_X \chi(x_2) q^{h w_2 x_2} e^{\left(2 x_2 + w_1 x + \frac{2 j w_1}{w_2} \right) (w_2 t)} d\mu_{-1}(x_1) \\
 &= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{w_2-1} (-1)^j \chi(j) q^{w_1 h j} T_{n, \chi, q^{w_2}}^{(h)} \left(w_1 x + \frac{2 j w_1}{w_2} \right) w_2^n \right) \frac{t^n}{n!}.
 \end{aligned} \tag{2.11}$$

By comparing coefficients $\frac{t^n}{n!}$ in the both sides of (2.10) and (2.11), we have the following theorem.

Theorem 2.4 *Let w_1 and w_2 be odd positive integers. Then we have*

$$\begin{aligned}
 &\sum_{j=0}^{w_1 d-1} (-1)^j \chi(j) q^{w_2 h j} T_{n, \chi, q^{w_1}}^{(h)} \left(w_2 x + \frac{2 j w_2}{w_1} \right) w_1^n \\
 &= \sum_{j=0}^{w_2 d-1} (-1)^j \chi(j) q^{w_1 h j} T_{n, \chi, q^{w_2}}^{(h)} \left(w_1 x + \frac{2 j w_1}{w_2} \right) w_2^n.
 \end{aligned}$$

If we take $x = 0$ in Theorem 2.4, we also derive the interesting identity for the generalized (h, q) -tangent numbers as follows:

$$\begin{aligned}
 &\sum_{j=0}^{w_1 d-1} (-1)^j \chi(j) q^{w_2 h j} T_{n, \chi, q^{w_1}}^{(h)} \left(\frac{2 j w_2}{w_1} \right) w_1^n \\
 &= \sum_{j=0}^{w_2 d-1} (-1)^j \chi(j) q^{w_1 h j} T_{n, \chi, q^{w_2}}^{(h)} \left(\frac{2 j w_1}{w_2} \right) w_2^n.
 \end{aligned}$$

Letting $q \rightarrow 1$ in Theorem 2.4, we can immediately have the generalized multiplication theorem for the generalized tangent polynomials (see, [3]). Observe that if $\chi = \chi^0$, then Theorem 2.4 reduces to Theorem 3.4 in [2].

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