Soliton Perturbation Theory for the Generalized Kawahara Equation

Anjan Biswas

Center for Research and Education in Optical Sciences and Applications
Department of Applied Mathematics and Theoretical Physics
Delaware State University, Dover, DE 19901-2277, USA
biswas.anjan@gmail.com

Essaid Zerrad

Department of Physics and Pre-Engineering
Center for Research and Education in Optical Sciences and Applications
Delaware State University, Dover, DE 19901-2277, USA

Abstract

The adiabatic parameter dynamics of solitons, due to the generalized Kawahara equation are obtained in this paper. The soliton perturbation theory is exploited to obtain the results. Also, the change in the velocity is obtained in presence of these perturbation terms.

1 INTRODUCTION

The dimensionless form of the generalized Kawahara equation (gKE) is given by

$$q_t + aq^p q_x + bq_{xxx} - cq_{xxxx} = 0$$

(1)
where $a$, $b$ and $c$ are constant parameters and $p > 0$. This equation is studied in the context of shallow water waves in fluid dynamics, particularly for $p = 1$ and $p = 2$.

It needs to be noted that (1) will fail the Painleve test of integrability for arbitrary $p$. Thus the classical methods of studying this gKE namely Inverse Scattering Transform, Backlund Transform or Hirota’s bilinear method will not work. However, (1) supports 1-soliton solution that is given by [1]

\[
q(x, t) = \frac{A}{\cosh^p B (x - \bar{x}(t))} \tag{2}
\]

where

\[
A = \left[ \frac{b}{ac} \frac{(p + 1)(p + 2)(p + 4)}{(p^2 + 4p + 8)^2} \right]^\frac{1}{p} \tag{3}
\]

\[
B = \frac{p}{2} \sqrt{\frac{b}{c(p^2 + 4p + 8)}} \tag{4}
\]

where $A$ is the amplitude of the soliton while $B$ is the inverse width of the soliton and $\bar{x}$ is the center position of the soliton. Thus, from (3) and (4), the amplitude and the inverse width are related as

\[
A^p = \frac{16cB^4}{ap^4} (p + 1)(p + 2)^2(p + 4) \tag{5}
\]

The velocity of the soliton is given by

\[
v = \frac{d\bar{x}}{dt} \tag{6}
\]

2 MATHEMATICAL PROPERTIES

Equation (1) has at least two integrals of motion that are known as linear momentum ($M$) and energy ($E$) [1, 4, 10]. These are respectively given by

\[
M = \int_{-\infty}^{\infty} q dx = \frac{A}{B} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{2}{p}\right)}{\Gamma\left(\frac{1}{2} + \frac{2}{p}\right)} \tag{7}
\]
and

\[ E = \int_{-\infty}^{\infty} q^2 dx = \frac{A^2}{B} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{4}{p}\right)}{\Gamma\left(\frac{1}{2} + \frac{4}{p}\right)} \]  

(8)

These conserved quantities are calculated by using the 1-soliton solution given by (2). Also, in (6) and (7), \( \Gamma(x) \) is the usual gamma function that is defined as

\[ \Gamma(x) = \int_{0}^{\infty} e^{-t} t^{x-1} dt \]  

(9)

The center of the soliton \( \bar{x} \) is given by the definition

\[ \bar{x} = \frac{\int_{-\infty}^{\infty} xq dx}{\int_{-\infty}^{\infty} q dx} = \frac{\int_{-\infty}^{\infty} xq dx}{M} \]  

(10)

where \( M \) is defined in (7). Thus, the velocity of the soliton is given by

\[ v = \frac{d\bar{x}}{dt} = \frac{\int_{-\infty}^{\infty} xq dx}{\int_{-\infty}^{\infty} q dx} = \frac{\int_{-\infty}^{\infty} xq dx}{M} \]  

(11)

On using (1), (2) and (6), the velocity of the soliton reduces to

\[ v = -\frac{4b^2}{c} \frac{p^2 + 4p + 4}{(p^2 + 4p + 8)^2} \]  

(12)

### 3 PERTURBATION TERMS

The perturbed fKdV equation that is going to be studied in this paper is given by

\[ q_t + aq^p q_x + bq_{xxx} - cq_{xxxx} = \epsilon R \]  

(13)

where, in (12), \( \epsilon \) is the perturbation parameter and \( 0 < \epsilon \ll 1 \), while \( R \) gives the perturbation terms. In presence of perturbation terms, the momentum and the energy of the soliton do not stay conserved. Instead, they undergo adiabatic changes that lead to the adiabatic deformation of the soliton amplitude, width and a slow change in the velocity [3]. Using (8), the law of adiabatic deformation of the soliton energy is given by

\[ \frac{dE}{dt} = 2\epsilon \int_{-\infty}^{\infty} qR dx \]  

(14)
while the adiabatic law of change of the velocity of the soliton, from (10), is
given by
\[ v = -\frac{4b^2}{c} \left( \frac{p^2 + 4p + 4}{(p^2 + 4p + 8)^2} \right) + \frac{\epsilon}{M} \int_{-\infty}^{\infty} x R dx \] (15)

In this paper, the perturbation terms that are going to be considered are
\[ R = \alpha q + \beta q_{xx} + \gamma q_x q_{xx} + \delta q^n q_x + \lambda q_{xxxx} + \nu q_x q_{xx} \]
\[ + \sigma q_x^3 + \xi q_x q_{xxxx} + \eta q_x q_{xxxx} + \rho q_{xxxx} + \psi q_{xxxxx} + \kappa q_{xxxxx} \] (16)

In \( R \), dissipation gives rise to the first two terms and so \( \alpha \) and \( \beta \) are small
dissipative coefficients [4]. Also, \( \delta \) or \( \psi \) represent the coefficient of higher order
nonlinear dispersive term [4] and \( m \) is a positive integer with \( 1 \leq m \leq 4 \) [7].
The coefficient of \( \rho \) provide a higher order stabilizing term and must therefore
be taken into account [4]. The perturbation term given by coefficient of \( \eta \) was
recently considered [6] while the remaining perturbation terms arise in the
context of extended version of integrable equations [7].

3.1 APPLICATIONS

In presence of these perturbation terms, the adiabatic variation of the energy
of the soliton is given by
\[ \frac{dE}{dt} = \frac{2\epsilon A^2}{Bp^2} \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{6}{p} \right) \frac{256\kappa(p^2 + 4p - 72)B^4}{p(p + 4)(p + 12)} \frac{\Gamma \left( \frac{6}{p} \right)}{\Gamma \left( \frac{1}{2} + \frac{6}{p} \right)} \]
\[ \frac{1}{p(p + 8)(3p + 8)} \{ \alpha p^2(p + 8)(3p + 8) - 16\beta B^2p(3p + 8) + 256\rho(3p + 3)B^4 \} \frac{\Gamma \left( \frac{4}{p} \right)}{\Gamma \left( \frac{1}{2} + \frac{4}{p} \right)} \] (17)

The law of the change of velocity for the given perturbation terms in (16) is
given by
\[ v = -\frac{2aA^2(p + 2)^2}{b(p^2 + 4p + 8)^2} + \frac{\epsilon}{\Gamma \left( \frac{2}{p} + \frac{1}{2} \right)} \left[ \frac{\delta A^m}{m + 1} \frac{\Gamma \left( \frac{2m+2}{p} \right)}{\Gamma \left( \frac{1}{2} + \frac{2m+2}{p} \right)} \right] \]
\[ + \frac{8AB^2}{p^2(p + 8)(3p + 8)} \{ p(3p + 8)(\gamma - 2\lambda) - 16B^2(p + 3)(3\xi - \eta) \} \frac{\Gamma \left( \frac{4}{p} \right)}{\Gamma \left( \frac{1}{2} + \frac{4}{p} \right)} \]
\[ 8A^2B^2 \left\{ \nu p - 2\sigma(p + 18) \right\} \frac{\Gamma \left( \frac{a}{p} \right)}{\Gamma \left( \frac{1}{2} + \frac{a}{p} \right)} \]

(18)

4 CONCLUSIONS

In this paper, soliton perturbation theory is used to study the gKE. In future, it is possible to extend these perturbation terms to include other perturbation terms that include the non-local ones too. The quasi-stationary aspects of the perturbed soliton in presence of such perturbation terms will be studied and reported in future publications.

ACKNOWLEDGEMENT

This research of the first author (AB) was fully supported by NSF-CREST Grant No: HRD-970668 and the support is very thankfully appreciated.

The research work of the second author is financially supported by the Army Research Office (ARO) under the award number W911NF-05-1-0451 and this support is thankfully appreciated.

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Received: December 21, 2007