Adv. Theor. Appl. Mech., Vol. 1, 2008, no. 7, 327 - 335

The Homotopy Analysis Method for Solving the Sawada–Kotera and Lax's Fifth-Order KdV Equations

Saeed Dinarvand [1,*], **Soroush Khosravi** [2],

Ahmad Doosthoseini [1], **and Mohammad Mehdi Rashidi** [1]

1. Mechanical Engineering Department, Engineering Faculty of Bu-Ali Sina University, Hamedan, Iran

2. Industrial Engineering Department, Mazandaran University of Science and Technology, Babol, Iran

Abstract

In this Letter, we apply the homotopy analysis method (HAM) to obtain approximate analytical solutions of the Sawada–Kotera and Lax's fifth-order KdV equations, without any linearization or weak nonlinearity assumptions. The homotopy analysis method contains the auxiliary parameter *h*, which provides us with a simple way to adjust and control the convergence region of solution series. This method provides an efficient approximate analytical solution with high accuracy, minimal calculation, avoidance of physically unrealistic assumptions.

Keywords: Homotopy analysis method; Generalized fifth-order KdV (gfKdV) equation; Sawada–Kotera equation; Lax's fifth-order KdV equation

1. Introduction

Nonlinear phenomena play a crucial role in applied mathematics and physics. In the past decades, both mathematicians and physicists have devoted considerable effort to the study of explicit solutions to nonlinear integer-order differential equation. Many powerful methods proposed to solve this equations. Among them, the homotopy analysis method (HAM) [1–6] provides an effective procedure for explicit and numerical solutions of a wide and general class of differential systems representing real physical problems. Based

* E-mail address: saeed_dinarvand@yahoo.com

on homotopy of topology, the validity of the HAM is independent of whether or not there exist small parameters in the considered equation. Therefore, the HAM can overcome the foregoing restrictions and limitations of perturbation techniques so that it provides us with a possibility to analyze strongly nonlinear problems. This method has been successfully applied to solve many types of nonlinear problems by others [7–13].

The main goal of this Letter is to extend the homotopy analysis method to solve fifth-order KdV (fKdV) equations, namely, Sawada–Kotera equation and Lax's fifth-order KdV equation.

Consider the generalized fifth-order KdV (gfKdV) equation

$$u_t + au^2u_x + bu_xu_{xx} + cuu_{xxx} + du_{xxxxx} = 0, \tag{1}$$

where a, b, c and d are constants. Equation (1) is the well-known the Sawada–Kotera equation if we set $a = 45$, $b = 15$, $c = 15$ and $d = 1$ [14] and the Lax's fifth-order KdV equation by setting $a = 30$, $b = 30$, $c = 10$ and $d = 1$.

In this Letter, the basic idea of the HAM is introduced and then its application in the Sawada–Kotera and Lax's fifth-order KdV equations is studied. In addition, comparison is made with the exact solution.

2. Basic idea of the HAM

Let us consider the following differential equation

$$\mathrm{N}\left[u(\tau)\right] = 0, \tag{2}$$

where N is a nonlinear operator, τ denotes independent variable, $u(\tau)$ is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao [3] constructs the so-called *zero-order deformation equation*

$$(1-p)\mathrm{L}\left[\varphi(\tau;p) - u_0(\tau)\right] = phH(\tau)\mathrm{N}\left[\varphi(\tau;p)\right], \tag{3}$$

where $p \in [0,1]$ is the embedding parameter, $h \neq 0$ is a non-zero auxiliary parameter, $H(\tau) \neq 0$ is an auxiliary function, L is an auxiliary linear operator, $u_0(\tau)$ is an initial guess of $u(\tau)$, $\varphi(\tau;p)$ is a unknown function, respectively. It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, when $p = 0$ and $p = 1$, it holds

$$\varphi(\tau;0) = u_0(\tau), \qquad \varphi(\tau;1) = u(\tau),$$

respectively. Thus as p increases from 0 to 1, the solution $\varphi(\tau;p)$ varies from the initial guess $u_0(\tau)$ to the solution $u(\tau)$. Expanding $\varphi(\tau;p)$ in Taylor series with respect to p, we have

$$\varphi(\tau;p) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau)p^m, \tag{4}$$

Where

$$u_m(\tau) = \frac{1}{m!} \frac{\partial^m \varphi(\tau;p)}{\partial p^m}\bigg|_{p=0}. \tag{5}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter *h*, and the auxiliary function are so properly chosen, the series (4) converges at $p=1,$ then we have

$$u(\tau) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau), \tag{6}$$

which must be one of solutions of original nonlinear equation, as proved by Liao [3]. As $h=-1$ and $H(\tau)=1$, equation (3) becomes

$$(1-p)\mathrm{L}\left[\varphi(\tau;p) - u_0(\tau)\right] + p\,\mathrm{N}\left[\varphi(\tau;p)\right] = 0, \tag{7}$$

which is used mostly in the homotopy perturbation method, where as the solution obtained directly, without using Taylor series [15,16].

According to the definition (5), the governing equation can be deduced from the *zero-order deformation equation* (3). Define the vector

$$\vec{u}_n = \{u_0(\tau), u_1(\tau), \ldots, u_n(\tau)\}.$$

Differentiating equation (3) *m* times with respect to the embedding parameter *p* and then setting $p=0$ and finally dividing them by *m*!, we have the so-called *mth-order deformation equation*

$$\mathrm{L}\left[u_m(\tau) - \chi_m u_{m-1}(\tau)\right] = hH(\tau)R_m(\vec{u}_{m-1}), \tag{8}$$

where

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1}\mathrm{N}\left[\varphi(\tau;p)\right]}{\partial p^{m-1}}\bigg|_{p=0}, \tag{9}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases}$$

It should be emphasized that $u_m(\tau)$ for $m \ge 1$ is governed by the linear equation (8) with the linear boundary conditions that come from original problem, which can be easily solved by symbolic computation software such as Maple and Mathematica.

3. Application

3.1. The generalized fifth-order KdV equation

In this section we apply the homotopy analysis method for the generalized fifth-order KdV (gfKdV) equation (1). We start with initial approximation $u_0(x,t) = u(x,0)$ and the linear operator

$$\mathrm{L}\left[\varphi(x,t;p)\right] = \frac{\partial \varphi(x,t;p)}{\partial t}, \tag{10}$$

possesses the property

$$\mathrm{L}(\mathrm{c}_1) = 0, \tag{11}$$

where c_1 is an integral constant to be determined by initial condition. Furthermore, equation (1) suggests to define the nonlinear operator

$$\begin{aligned} \mathrm{N}\left[\varphi(x,t;p)\right] = & \frac{\partial\varphi(x,t;p)}{\partial t} + a(\varphi(x,t;p))^2 \frac{\partial\varphi(x,t;p)}{\partial x} \\ & + b\frac{\partial\varphi(x,t;p)}{\partial x}\frac{\partial^2\varphi(x,t;p)}{\partial x^2} + c\,\varphi(x,t;p)\frac{\partial^3\varphi(x,t;p)}{\partial x^3} + d\frac{\partial^5\varphi(x,t;p)}{\partial x^5}. \end{aligned} \tag{12}$$

Using the above definition, with assumption $H(\tau)=1,$ we construct the *zero-order deformation equation*

$$(1-p)\mathrm{L}\left[\varphi(x,t;p) - u_0(x,t)\right] = ph\,\mathrm{N}\left[\varphi(x,t;p)\right]. \tag{13}$$

Obviously, when $p=0$ and $p=1$,

$$\varphi(x,t;0) = u_0(x,t), \qquad \varphi(x,t;1) = u(x,t). \tag{14}$$

Differentiating the *zero-order deformation equation* (13) *m* times with respect to *p*, and finally dividing by *m*!, we have the *mth-order deformation equation*

$$\mathrm{L}\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] = hR_m(\vec{u}_{m-1}), \tag{15}$$

subject to initial condition

$$u_m(x,0) = 0, \tag{16}$$

where

$$\begin{aligned} R_m(\vec{u}_{m-1}) = & \frac{\partial u_{m-1}(x,t)}{\partial t} + a\sum_{n=0}^{m-1}\left(\sum_{j=0}^{n} u_j(x,t)u_{n-j}(x,t)\right)\frac{\partial u_{m-1-n}(x,t)}{\partial x} \\ & + b\sum_{n=0}^{m-1}\frac{\partial u_n(x,t)}{\partial x}\frac{\partial^2 u_{m-1-n}(x,t)}{\partial x^2} + c\sum_{n=0}^{m-1} u_n(x,t)\frac{\partial^3 u_{m-1-n}(x,t)}{\partial x^3} + d\frac{\partial^5 u_{m-1}(x,t)}{\partial x^5} \end{aligned} \tag{17}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases}$$

Obviously, the solution of the *mth-order deformation equation* (15) for $m \ge 1$ becomes

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + h\mathrm{L}^{-1}\left[R_m(\vec{u}_{m-1})\right]. \tag{18}$$

In the following parts, we apply the homotopy analysis method to solve the Sawada–Kotera and Lax's fifth-order KdV equations.

3.1.1. The Sawada–Kotera equation

Consider the gfKdV equation (1), for $a=45,\ b=15,\ c=15$ and $d=1,$ in this case that is, the Sawada–Kotera equation. We choose the initial approximation [14]

$$u_0(x,t) = u(x,0) = 2\alpha^2 \operatorname{sech}^2[\alpha(x-\lambda)], \tag{19}$$

where α and λ are arbitrary constants and $\alpha \neq 0$. Using previous formulae to determine other components of the solution series. From (18) and (19), we have

$$u_1(x,t) = -64h\alpha^7 t \operatorname{sech}^2[\alpha(x-\lambda)]\tanh[\alpha(x-\lambda)], \tag{20}$$

$$u_2(x,t) = 32h\alpha^7 t \operatorname{sech}^4[\alpha(x-\lambda)]\big(-32h\alpha^5 t + 16h\alpha^5 t \cosh[2\alpha(x-\lambda)] - (1+h)\sinh[2\alpha(x-\lambda)]\big), \tag{21}$$

$$\begin{aligned} u_3(x,t) = -\frac{16}{3}h\alpha^7 t \operatorname{sech}^5[\alpha(x-\lambda)]\big(&288h(1+h)\alpha^5 t \cosh[\alpha(x-\lambda)] - 96h(1+h)\alpha^5 t \cosh[\alpha(x-\lambda)] \\ &+ 2\big(3+6h+h^2(3-2560\alpha^{10}t^2) + (3+6h+h^2(3+512\alpha^{10}t^2))\cosh[2\alpha(x-\lambda)]\big)\sinh[\alpha(x-\lambda)]\big), \end{aligned} \tag{22}$$

$$\begin{aligned} u_4(x,t) = -\frac{8}{3}h\alpha^7 t \operatorname{sech}^6[\alpha(x-\lambda)]\big(&32h\alpha^5 t(9+18h+h^2(9+3328\alpha^{10}t^2))\cosh[2\alpha(x-\lambda)] \\ &-16h\alpha^5 t(9+18h+h^2(9+256\alpha^{10}t^2))\cosh[4\alpha(x-\lambda)] + 3\big(16h\alpha^5 t(9+18h+h^2(9-2816\alpha^{10}t^2)) \\ &-2(1+h)(-1-2h+h^2(-1+2560\alpha^{10}t^2))\sinh[2\alpha(x-\lambda)] + (1+h)(1+2h+h^2(1+512\alpha^{10}t^2))\sinh[4\alpha(x-\lambda)]\big)\big), \end{aligned} \tag{23}$$

$$\vdots$$

We used 10 terms in evaluating the approximate solution $u_{app} = \sum_{i=0}^{9} u_i$. Note that the solution series contains the auxiliary parameter h which provides us with a simple way to adjust and control the convergence of the solution series. In general, by means of the so-called *h-curve* i.e., a curve of a versus h. As pointed by Liao [3], the valid region of h is a horizontal line segment. Therefore, it is straightforward to choose an appropriate range for h which ensure the convergence of the solution series. We stretch the *h-curve* of $u_{ttt}(0,0)$ in Fig. 1, which shows that the solution series is convergent when $-1.35 < h < -0.65$.

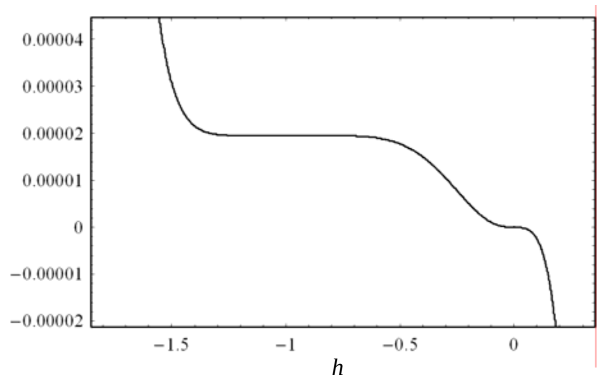


Fig. 1. The *h-curve* of $u_{ttt}(0,0)$ given by the 10th-order approximate solution, when $\alpha = 0.3$ and $\lambda = 0.4$.

In continuation, we take the middle value $h = -1$ and compute the absolute errors for differences between the exact solution $u(x,t) = 2\alpha^2 \operatorname{sech}^2[\alpha(x - 16\alpha^4 t - \lambda)]$ and the approximate solution at some points. The results are listed in Table 1. The behavior of the solution obtained by HAM and exact solution are shown in Fig. 2(a,b). Comparison of the result obtained by HAM with exact solution reveals that the accuracy of the new method.

Table 1
Absolute errors for differences between the exact solution and the 10th-order approximate solution given by HAM for $h = -1$, when $\alpha = 0.3$ and $\lambda = 0.4$.

x	t			
	1	2	3	4
1	2.48510×10^{-16}	3.33344×10^{-14}	2.11892×10^{-12}	4.11389×10^{-11}
5	1.38778×10^{-17}	3.25343×10^{-15}	1.87218×10^{-13}	3.30741×10^{-12}
10	1.30104×10^{-18}	1.99493×10^{-17}	1.18872×10^{-15}	2.13566×10^{-14}
15	8.13152×10^{-20}	1.62630×10^{-19}	1.01915×10^{-17}	1.82905×10^{-16}
20	5.08220×10^{-21}	4.23516×10^{-21}	7.29295×10^{-19}	1.32628×10^{-17}
25	1.58819×10^{-22}	3.17637×10^{-22}	3.70048×10^{-20}	6.72491×10^{-19}

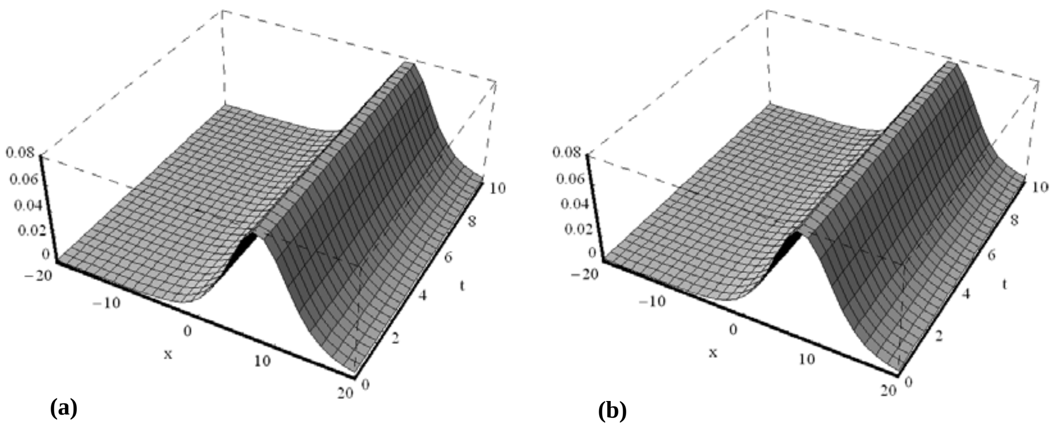


Fig. 2. The behavior of the solutions obtained by: (a) HAM for $h = -1$; (b) exact solution, when $\alpha = 0.2$ and $\lambda = 8$.

3.1.2. The Lax's fifth-order KdV equation

Consider the gfKdV equation (1), for $a = 30$, $b = 30$, $c = 10$ and $d = 1$, in this case that is, the *Lax's* fifth-order KdV equation. We choose the initial approximation [14]

$$u_0(x,t) = u(x,0) = 2\alpha^2(2 - 3\tanh^2[\alpha(x-\lambda)]), \tag{24}$$

where α and λ are arbitrary constants and $\alpha \neq 0$. Using previous formulae to determine other components of the solution series. From (18) and (24), we have

$$u_1(x,t) = -6h\alpha^7 t\,\mathrm{sech}^7[\alpha(x-\lambda)](-586\sinh[\alpha(x-\lambda)] + 141\sinh[3\alpha(x-\lambda)] + 7\sinh[5\alpha(x-\lambda)]), \quad (25)$$

$$\begin{aligned} u_2(x,t) = \frac{3}{16}h\alpha^7 t\,\mathrm{sech}^{12}[\alpha(x-\lambda)]\big(&36473952h\alpha^5 t - 44140272h\alpha^5 t\cosh[2\alpha(x-\lambda)] \\ &+ 9742464h\alpha^5 t\cosh[4\alpha(x-\lambda)] - 853656h\alpha^5 t\cosh[6\alpha(x-\lambda)] + 28448h\alpha^5 t\cosh[8\alpha(x-\lambda)] \\ &+ 392h\alpha^5 t\cosh[10\alpha(x-\lambda)] + 1626(1+h)\sinh[2\alpha(x-\lambda)] + 864(1+h)\sinh[4\alpha(x-\lambda)] \\ &- 189(1+h)\sinh[6\alpha(x-\lambda)] - 176(1+h)\sinh[8\alpha(x-\lambda)] - 7(1+h)\sinh[10\alpha(x-\lambda)]\big), \end{aligned} \quad (26)$$

$$\vdots$$

We used 10 terms in evaluating the approximate solution $u_{app} = \sum_{i=0}^{9} u_i$ and stretch the *h-curve* of $u_{tt}(0,0)$ in Fig. 3, which shows that the solution series is convergent when $-1.4 < h < -0.4$. The absolute errors for differences between the exact solution $u(x,t) = 2\alpha^2(2 - 3\tanh^2[\alpha(x - 56\alpha^4 t - \lambda)])$ and the approximate solution for $h = -0.7$ at some points are listed in Table 2. Also, to verify how much the approximate solution is accurate, we show the behavior of the approximate solution and the exact solution in Fig. 4(a,b).

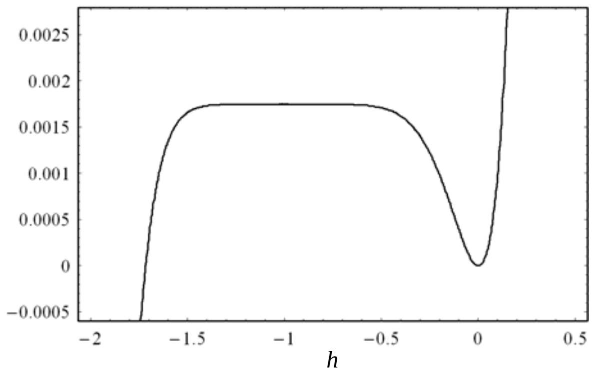
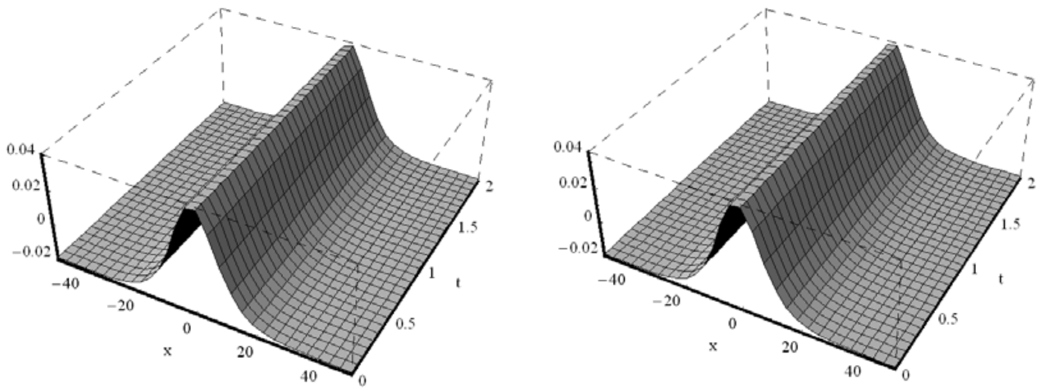


Fig. 3. The *h-curve* of $u_{tt}(0,0)$ given by the 10th-order approximate solution, when $\alpha = 0.2$ and $\lambda = 0.3$.

Table 2

Absolute errors for differences between the exact solution and the 10th-order approximate solution given by HAM for $h = -0.7$, when $\alpha = 0.2$ and $\lambda = 0.3$.

x	t			
	1	2	3	4
15	4.70936×10^{-7}	8.73116×10^{-7}	5.28097×10^{-6}	6.73880×10^{-5}
25	1.60890×10^{-9}	3.42460×10^{-9}	5.45908×10^{-9}	7.71694×10^{-9}
35	3.59546×10^{-13}	1.33804×10^{-12}	3.16755×10^{-12}	6.12840×10^{-12}
45	1.66256×10^{-14}	4.60326×10^{-14}	9.26897×10^{-14}	1.61898×10^{-13}



(a) (b)

Fig. 4. The behavior of the solutions obtained by: (a) HAM for $\hbar = -0.6$; (b) exact solution, when $\alpha = 0.1$ and $\lambda = 0.2$.

4. Conclusions

In this Letter we solved the Sawada–Kotera and Lax's fifth-order KdV equations by the homotopy analysis method. The HAM provides us with a convenient way to control the convergence of approximation series, which is a fundamental qualitative difference in analysis between HAM and other methods. This method dose not require small parameter in any equation as same as the perturbation approach. The numerical results of the above problems display a fast convergence, with minimal calculations. It shows that the HAM is a very efficient method. We sincerely hope this method can be applied in a wider range.

References

[1] S.J. Liao, The proposed homotopy analysis technique for the solution of nonlinear problems, PhD thesis, Shanghai Jiao Tong University, 1992.

[2] S.J. Liao, An explicit, totally analytic approximation of Blasius viscous flow problems, Int. J. Nonlinear Mech. 34 (1999) 759–778.

[3] S.J. Liao, Beyond Perturbation: Introduction to the Homotopy Analysis Method, Chapman & Hall/CRC Press, Boca Raton, 2003.

[4] S.J. Liao, On the analytic solution of magnetohydrodynamic flows of non-Newtonian fluids over a stretching sheet, J. Fluid Mech. 488 (2003) 189–212.

[5] S.J. Liao, On the homotopy analysis method for nonlinear problems, Appl. Math. Comput. 147 (2004) 499–513.

[6] S.J. Liao, A new branch of solutions of boundary-layer flows over an impermeable stretched plate, Int. J. Heat Mass Transfer 48 (2005) 2529.

[7] S.J. Liao, Comparison between the homotopy analysis method and homotopy perturbation method, Appl. Math. Comput. 169 (2005) 1186–1194.

[8] S.J. Liao, J. Su, A.T. Chwang, Series solutions for a nonlinear model of combined convective and radiative cooling of a spherical body, Int. J. Heat Mass Transfer 49 (2006) 2437.

[9] S.J. Liao, E. Magyari, Z. Angew. Math. Phys. 57 (2006) 777.

[10] S.J. Liao, Stud. Appl. Math. 117 (2006) 239.

[11] S. Abbasbandy, The application of homotopy analysis method to nonlinear equations arising in heat transfer, Phys. Lett. A 360 (2006) 109–113.

[12] S.P. Zhu, ANZIAM J. 47 (2006) 477.

[13] S. Abbasbandy, The application of homotopy analysis method to solve a generalized Hirota–Satsuma coupled KdV equation, Phys. Lett. A 361 (2007) 478–483.

[14] Parkes EJ, Duffy BR. An automated Tanh-function method for finding solitary wave solitons to non-linear evolution equations.Comput PhysCommun 1996;98(3):288–300.

[15] J.H. He, Homotopy perturbation method for solving boundary value problems, Phys. Lett. A 350 (2006) 87–88.

[16] J.H. He, A coupling method for homotopy technique and perturbation technique for non-linear problem, Int. J. Nonlinear. Mech. 35 (2000) 37–43.

Received: November, 2008