A Variation Problem for Null Curves

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Abstract


Considerable works were done on physical and geometric use of null curves in the literature ([2]-[9]). In this work, we give the analogue of the Frenet -Serret formulas on the lightlike surface and we derive the formula square curvature for a non-unit velocity null curve in $\mathbb{E}^3_1$. We obtain the intrinsic equations for a relaxed elastic line on a lightlike surface in the 3-dimensional semi-Euclidean space

Keywords: Variational problem, null curve, lightlike surface

1 Preliminaries

In this section, we will give some new definitions and propositions except fundamental definitions in literature.

Definition 1 Let $\mathbb{E}^3 = \{(x^1, x^2, x^3, x^4)|x^1, x^2, x^3 \in \mathbb{R}\}$ be a 3-dimensional Euclidean space. For any vectors $x = (x^1, x^2, x^3), \quad y = (y^1, y^2, y^3) \in \mathbb{E}^3$, the pseudo scalar product of $x$ and $y$ is defined to be $\langle x, y \rangle = -x^1y^1 + x^2y^2 + x^3y^3 \in \mathbb{E}^3_1$ is called 3-dimensional semi-Euclidean space $\mathbb{E}^3_1$ [10].

Definition 2 Let $\mathbb{E}^3_1$ be a semi-Euclidean space furnished with a metric tensor $\langle , \rangle$. A vector $v$ to $\mathbb{E}^3_1$ is called if $\langle v, v \rangle = 0$ and $v \neq 0, v$ is null [10]

Proposition 3 Let $C : [a, b] \to \mathbb{E}^3_1$, a positively oriented set $\{C'(p), C''(p), C'''(p)\}$, there exist a local frame $F = \{\xi = C', N, W\}$, called Cartan frame satisfying $\langle \xi, \xi \rangle = \langle N, N \rangle = 0, \langle \xi, N \rangle = 1, \langle W, \xi \rangle = \langle W, N \rangle = 0, \langle W, W \rangle = 1$. The Cartan equations $\xi' = \kappa W, N' = \tau W, W' = \tau \xi - \kappa N$ where $\kappa$ and $\tau$, the curvature and torsion of $C$ ([2] - [9]).
Definition 4 Let $\xi$ denote an arc on a connected oriented lightlike surface $L$ parametrized by arc length $s$, $0 \leq s \leq l$. Let $\kappa(s)$ be the curvature of $\xi(s)$. The total square curvature $C$ of $\xi$ is defined by $C = \int_0^l \kappa^2 ds$.

Definition 5 The curve null $\xi$ is called a relaxed elastic line if it is an extremal for the variational problem of the minimizing value of $L$ within the family of all curves of length $l$ on $L$ lightlike surface having the same initial point and initial direction with $\xi$.

Definition 6 Let $\xi$ be a null curve on the lightlike surface $L$. Apart from the Frenet frame, there also exist a second frame at every point of the null curve $\xi$. At the point $\xi(s)$, let $T(s) = \xi'(s)$ denote the unit tangent vector to $\xi$, let $N(s)$ denote the unit normal to $L$ lightlike surface. We take $T \times N = -T$, $T \times R = -N$, $N \times R = -R$. Then, $\{T, R, N\}$ gives an orthonormal basis for all vectors at $\xi(s)$.

Proposition 7 Let $L$ be the lightlike surface and $\xi$ denote a null curve on $L$. We give the analogue of the Frenet-Serret formulas on a lightlike surface in $E_3^1$.

$$
\begin{bmatrix}
T' \\
R' \\
N'
\end{bmatrix} = \begin{bmatrix}
k_n & k_g & 0 \\
k_g & 0 & -k_g \\
0 & -\tau_g & -k_n
\end{bmatrix} \begin{bmatrix}
T \\
R \\
N
\end{bmatrix}
$$

(1)

$k_g$ is the geodesic curvature, $\tau_g$ is the geodesic torsion, $k_n$ is the normal curvature. We can write

$$
T' = a_{11}T + a_{12}R + a_{13}N,
$$

$$
R' = a_{21}T + a_{22}R + a_{23}N,
$$

$$
N' = a_{31}T + a_{32}R + a_{33}N
$$

$\langle T, T \rangle = \langle N, N \rangle = 0, \langle T, N \rangle = 1, \langle R, T \rangle = \langle R, N \rangle = 0, \langle R, R \rangle = 1$ and $< T', T > = 0 \Rightarrow a_{13} = 0$, $< R', R >= 0 \Rightarrow a_{22} = 0$ and $< N', N > = 0 \Rightarrow a_{31} = 0$ $< T', R >= k_g \Rightarrow a_{12} = k_g$ and $< R', T >= a_{23} = -k_g$, $< T', N >= k_n \Rightarrow a_{11} = k_n$; $< R', N >= \tau_g \Rightarrow a_{21} = \tau_g; < N', R >= -\tau_g \Rightarrow a_{32} = -\tau_g$.

With substituting $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}$ in (2), we obtain (1). 

Proposition 8 Let $\gamma : I \rightarrow E_3^1$ denote a non-unit velocity null curve. The square curvature $\kappa^2$ along the line is given $\kappa^2 = \left( \frac{\partial^2}{\partial s^2} \gamma, \frac{\partial^2}{\partial s^2} \gamma \right)^{-1}$. 

Proof. With the first and third differentiation of $\gamma$, respectively, we obtain

$$\frac{\partial \gamma}{\partial \sigma} = \frac{d^2 \gamma}{ds \, ds} = T \left( \frac{\partial^2 \gamma}{\partial \sigma^2} \right)^{1/4}$$

(3)

$$\frac{\partial^3 \gamma}{\partial \sigma^3} = \left( \frac{\partial^3 \gamma}{\partial \sigma^3} \right)^{1/4} - \kappa \tau \left( \frac{\partial^2 \gamma}{\partial \sigma^2} \right)^{1/4} \frac{d}{ds} + \left[ \kappa \left( \frac{\partial^2 \gamma}{\partial \sigma^2} \right)^{1/4} d \left( \frac{\partial^2 \gamma}{\partial \sigma^2} \right)^{1/4} \right] \frac{\partial \gamma}{\partial \sigma}$$

$$+ 2\kappa \left( \frac{\partial^2 \gamma}{\partial \sigma^2} \right)^{1/4} d \left( \frac{\partial^2 \gamma}{\partial \sigma^2} \right)^{1/4} \frac{\partial \gamma}{\partial \sigma} \right) W + \kappa^2 \left( \frac{\partial^2 \gamma}{\partial \sigma^2} \right)^{1/4} N.$$  

(4)

From $\left( \frac{\partial \gamma}{\partial \sigma}, \frac{\partial^3 \gamma}{\partial \sigma^3} \right) = \kappa^2 \left( \frac{\partial^2 \gamma}{\partial \sigma^2} \right)$, proof is trivial.

2 The solution of $C'(0) = 0$

Let $\xi$ null curve be lies in a coordinate patch $(u, v) \to r(u, v)$ of lightlike surface $L$. Let be $r_u = \frac{\partial r}{\partial u}$, $r_v = \frac{\partial r}{\partial v}$. Then, $\xi$ is expressed as $\xi(s) = r(u(s), v(s))$, $0 \leq s \leq l$ with $T(s) = \xi'(s) = \frac{d}{ds} r_u + \frac{d}{ds} r_v$ and $R(s) = r_v r_u + y(s) r_v$ for scalar functions $u(s)$ and $y(s)$.

We extend $\xi$ to an arc $\xi^*$ defined for $0 \leq s \leq l^*$, with $l^* > l$, $l^*$ close to $l$. $\mu(s)$, $0 \leq s \leq l^*$, is not vanishing identically. Define $\eta(s) = \mu(s) v^*(s)$, $\rho(s) = \mu(s) y^*(s)$ Along $\xi$

$$\mu(0) = 0, \mu'(0) = 0$$

(5)

Define $\eta(s)r_u + \rho(s) r_v = \mu(s) R(s)$ and

$$\eta(\sigma; t) = r_u(\sigma) + t \rho(\sigma) + v(\sigma) + t \nu(\sigma)$$

(6)

For $|t| < \varepsilon$ (where $\varepsilon > 0$ depends upon the choice of $\xi^*$ and of $\mu$), the point $\eta(\sigma; t)$ lies in the coordinate patch. For fixed $t$, $\eta(\sigma; t)$ gives an arc with the same initial point and initial direction as $\xi$. $\sigma$ is not arc length for $t \neq 0$. For fixed $t$, $|t| < \varepsilon$, let $J^*(t)$ denote the length of the arc $\eta(\sigma; t), 0 \leq \sigma \leq l^*$. Then $J^*(t) = \int_0^{l^*} \left( \frac{\partial^2 \eta}{\partial \sigma^2} (\sigma; t) \right)^{1/4} d\sigma$. $J^*(0) = l^* > l$. For $0 < \varepsilon_1 \leq \varepsilon$, $J^*(t) > l$ ($|t| < \varepsilon_1$). We can restrict $\eta(\sigma; t), 0 \leq |t| < \varepsilon_1$, to an arc of length $l$ by restricting the parameter $\sigma$ to an interval $0 \leq \sigma \leq \lambda(t) \leq l^*$. 

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\[ \int_0^\lambda(t) \left\langle \frac{\partial^2 \eta}{\partial \sigma^2} (\sigma; t), \frac{\partial^2 \eta}{\partial \sigma^2} (\sigma; t) \right\rangle^{1/4} d\sigma = l. \] (7)

For \( t = 0 \), if \( \lambda(t) \) differentiate with respect to \( t \), we obtain

\[
\frac{d\lambda}{dt} \bigg|_{t=0} = \frac{1}{2} k_g^{-1/2} l \int_0^t \left( \frac{\partial^2 \eta}{\partial \sigma^2} \bigg|_{t=0}, \frac{\partial^2 \eta}{\partial \sigma^2} \bigg|_{t=0} \right)^{-3/4} \left( \frac{\partial^3 \eta}{\partial t \partial \sigma^2} \bigg|_{t=0}, \frac{\partial^3 \eta}{\partial t \partial \sigma^2} \bigg|_{t=0} \right) d\sigma = 0
\] (8)

\[
\left. \frac{\partial \eta}{\partial \sigma} \right|_{t=0} = T, \quad 0 \leq \sigma \leq l ,
\] (9)

\[
\left. \frac{\partial^2 \eta}{\partial \sigma^2} \right|_{t=0} = T' = k_n T + k_g R ,
\] (10)

\[
\left. \frac{\partial^3 \eta}{\partial \sigma^3} \right|_{t=0} = T' = (k_n + k_n^2 + k_g k_n) T + (k_n k_g + k_n^2) R - k_g N.
\] (11)

From (6),

\[
\left. \frac{\partial \eta}{\partial t} \right|_{t=0} = \mu R .
\] (12)

We obtain respectively (13), (14), (15), with second, third and fourth differentiation of (12)

\[
\frac{\partial^2 \eta}{\partial t \partial \sigma} = \mu' R + \mu \tau_g T - \mu k_g N ,
\] (13)

\[
\left. \frac{\partial^3 \eta}{\partial t \partial \sigma^2} \right|_{t=0} = (2\mu' \tau_g + \mu \tau_g' + \mu \tau_g k_n) T + (\mu'' + 2\mu \tau_g k_n) R + (\mu k_g k_n - \mu k_g' - 2\mu' k_g) N ,
\] (14)
\[ \frac{\partial^4 \eta}{\partial t \partial \sigma^3} \bigg|_{t=0} = \left( 3 \mu'' \tau_k + 3 \mu' \tau'_k + \mu' \tau''_k + 3 \mu' \tau_k k_n + 2 \mu' \tau_k k'_n + \mu \tau_k k'_n + \mu \tau_k k''_n + 2 \mu k^2 \right) T \]
\[ + \left( 6 \mu' \tau_k k_g + 3 \mu' \tau k'_g + \mu'' + 3 \mu k'_g \tau_g \right) R \]
\[ + \left( 3 \mu' k_g k_n - 3 \mu'' k_g + 2 \mu k'_g k_n - 2 \mu \tau_g k^2_g - 3 \mu k' k_g - \mu k_g k'_n - \mu k_g k''_n \right) N. \]

From Theorem 1.3, the total square curvature of the curve \( \eta(\sigma; t) \), \( 0 \leq \sigma \leq \lambda(t) \), \( t \neq 0 \).
\[ C(t) = \int_0^{\lambda(t)} \left\{ \langle \frac{\partial \eta}{\partial \sigma}, \frac{\partial^3 \eta}{\partial \sigma^3} \rangle \langle \frac{\partial^2 \eta}{\partial \sigma^2}, \frac{\partial \eta}{\partial \sigma^3} \rangle \langle \frac{\partial^2 \eta}{\partial \sigma^2}, \frac{\partial \eta}{\partial \sigma^2} \rangle \right\}^{-3/4} \, d\sigma. \]
If \( \xi \) extremal. \( C'(0) \) is equal zero. In calculating of \( C'(t) \),
\[ C'(t) = \frac{d\lambda}{dt} \left\{ \langle \frac{\partial \eta}{\partial \sigma}, \frac{\partial^3 \eta}{\partial \sigma^3} \rangle \langle \frac{\partial^2 \eta}{\partial \sigma^2}, \frac{\partial \eta}{\partial \sigma^3} \rangle \langle \frac{\partial^2 \eta}{\partial \sigma^2}, \frac{\partial \eta}{\partial \sigma^2} \rangle \right\}^{-3/4} \int_0^{\lambda(t)} \left\{ \langle \frac{\partial \eta}{\partial \sigma}, \frac{\partial^3 \eta}{\partial \sigma^3} \rangle \langle \frac{\partial^2 \eta}{\partial \sigma^2}, \frac{\partial \eta}{\partial \sigma^3} \rangle \langle \frac{\partial^2 \eta}{\partial \sigma^2}, \frac{\partial \eta}{\partial \sigma^2} \rangle \right\}^{-3/4} \, d\sigma \left[ \right. \]
\[ - \frac{3}{2} \int_0^{\lambda(t)} \left\{ \langle \frac{\partial \eta}{\partial \sigma}, \frac{\partial^3 \eta}{\partial \sigma^3} \rangle \langle \frac{\partial^2 \eta}{\partial \sigma^2}, \frac{\partial \eta}{\partial \sigma^2} \rangle \langle \frac{\partial^2 \eta}{\partial \sigma^2}, \frac{\partial \eta}{\partial \sigma^2} \rangle \right\}^{-7/4} \, d\sigma + \ldots \]

With using (8),(10), (11) , (13), (14), (15) in , (16), we find
\[ C'(0) = \int_0^l \mu'' k^{-1/2}_g \, ds - \int_0^l \mu' \left( 2k_n + 2k'_n k^-1_g \right) k^{-1} \, ds \]
\[ + \int_0^l \mu \left( \tau - k'' k^{-2}_g + k'_n k^{-1}_g + \frac{1}{2} k^2 k^{-1}_n - \frac{3}{2} k'_n k^{-2}_g \right) \, ds, \]

With integration by parts and (5),
\[ \int_0^l \mu'' k^{-1/2}_g \, ds = \mu'(l) k^{-1/2}_g(l) + \frac{1}{2} \mu(l) k'_g(l) k^{-3/2}_g(l) - \frac{1}{2} \int_0^l \mu \left( \frac{3}{2} k^{-5/2}_g(k'^{'}_g)^2 \right) ds \]
\[ \int_0^l \mu' (2k_n + 2k_g'k_g^{-1})k_g^{-1}ds = 2\mu(l)(k_n(l)k_g^{-1}(l) + k_g'(l)k_g^{-2}(l)) \] (19)

\[-2\int_0^l \mu(k_n'k_g^{-1} - k_nk_g^{-2}k_g' + k_g''k_g^{-2} - 2(k_g')^2k_g^{-3})ds \]

Substituting (18) in (17), we find

\[ C'(0) = \int_0^l \mu \left( \frac{\tau_g - k_n'k_g^{-2} + k_nk_g^{-1} + \frac{1}{2}k_n^2k_g^{-1} - \frac{3}{2}k'_gk_nk_g^{-2} - \frac{1}{2}k''_gk_g^{-3/2} + \frac{3}{2}k_g^{-5/2}(k_g')^2 + 2k_n'k_g^{-1} - k_nk_g^{-2}k_g' + k_g''k_g^{-2} - 2(k_g')^2k_g^{-3}}{k_g^{-2} - 2k_g'k_g^{-3/2} + \frac{1}{4}k_g'(l)k_g^{-3/2}(l) + \mu'(l)k_g^{-1/2}(l)} \right) ds \]

\[ C'(0) = 0 \] for all choices of the function \( \mu(s) \) satisfying (5), with arbitrary values of \( \mu(l) \) and \( \mu'(l) \), the given null arc \( \xi \) must satisfy two boundary conditions and differential equation

\[ \begin{align*}
1 & \quad k_g^{-1/2}(l) = 0 \\
2 & \quad k_g'(l) = -k_n(l)k_g(l) \\
3 & \quad \tau_g - k_n'k_g^{-2} + k_nk_g^{-1} + \frac{1}{2}k_n^2k_g^{-1} - \frac{3}{2}k'_gk_nk_g^{-2} - \frac{1}{2}k''_gk_g^{-3/2} + \frac{3}{2}k_g^{-5/2}(k_g')^2 + 2k_n'k_g^{-1} - k_nk_g^{-2}k_g' + k_g''k_g^{-2} - 2(k_g')^2k_g^{-3} = 0 .
\end{align*} \]

REFERENCES


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