Motion of a Bead on a Cycloid

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Abstract

The paper explains the theory behind time taken by a falling bead on a cycloid. The motion requires the path traveled by the bead from a higher point A to a lower point B along the cycloid. We will show that the time to fall from the point A to B on the curve given by the parametric equations $x = a(\theta - \sin \theta)$ and $y = a(\theta - \cos \theta)$ is independent of the starting point A.

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1. Introduction

Mathematics has been repeatedly applied to solve real life problems. The importance of mathematics cannot be seen as over emphasized because it is required in almost every field and career. One of the most important aspects of mathematics is geometry. The design and production of most technology require the application of geometry and this is proof of its importance [1].

A Cycloid curve which is generated by a point on a circle's circumference rolling on a plane is brachistochronous, because it represents the path completed in the shortest time between two points A and B for a given type of motion (such as a fall under the effect of gravity). The brachistochronous property of the cycloid was demonstrated by Jacques Bernoulli (1654-1705) in 1697. When comparisons are made between time taken traveling through an arc and that through a cycloid, [2,3,4] experiments and calculations show that it is faster by four hundreds of a second. In this paper we will establish the fact the time taken to fall from a point A to B on a cycloid is independent of the initial point A.

2. Mechanics of the Motion

The parametric equations of the cycloid are,

\[
\begin{align*}
x &= \sigma (\theta - \sin \theta) \\
y &= \sigma (1 - \cos \theta).
\end{align*}
\]

The derivatives of these equations are

\[
\begin{align*}
x' &= \sigma (1 - \cos \theta) \\
y' &= \sigma \sin \theta,
\end{align*}
\]

and

\[
\begin{align*}
x'^2 + y'^2 &= \sigma^2 \left( [1 - 2 \cos \theta + \cos^2 \theta] + \sin^2 \theta \right) \\
&= 2 \sigma^2 (1 - \cos \theta).
\end{align*}
\]

Using the conservation of energy law, we can obtain,
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\[ \frac{d}{2} m v^2 = m g y \]  

(7)

\[ v = \frac{ds}{dt} = \sqrt{2 g y} \]  

(8)

\[ dt = \frac{ds}{\sqrt{2 g y + d^2}} \]  

(9)

\[ \sqrt{2} g y \]  

(10)

\[ \frac{\sqrt{2} g y}{\alpha (1 - \cos \theta) d \theta} \]  

(11)

\[ \sqrt{\frac{\alpha}{g}} d \theta. \]  

(12)

So the time required to travel from the top to the bottom of the cycloid is,

\[ T = \int_{0}^{\pi} dt = \sqrt{\frac{\alpha}{g}}. \]  

(13)

However, from an intermediate point \( \theta_0 \), say C,

\[ v = \frac{ds}{dt} = \sqrt{2 g (y - y_0)} \]  

(14)

so

\[ T = \int_{\theta_0}^{\pi} \sqrt{\frac{2 \alpha^2 (1 - \cos \theta)}{2 m g (\cos \theta_0 - \cos \theta)}} d \theta \]  

(15)

\[ = \sqrt{\frac{\alpha}{g}} \int_{\theta_0}^{\pi} \sqrt{\frac{1 - \cos \theta}{\cos \theta_0 - \cos \theta}} d \theta. \]  

(16)

To integrate, we rearrange this equation and use half-angle formulas,
\[
\sin \left( \frac{t}{2} \right) = \sqrt{\frac{1 - \cos x}{2}} \tag{17}
\]
\[
\cos \left( \frac{t}{2} \right) = \sqrt{\frac{1 + \cos x}{2}} \tag{18}
\]

with the latter rewritten in the form,
\[
\cos \theta = 2 \cos^2 \left( \frac{t}{2} \right) - 1 \tag{19}
\]

to obtain,
\[
T = \sqrt{\frac{\alpha}{\varepsilon}} \int_{\theta_0}^{\theta} \sin \left( \frac{t}{2} \theta \right) d\theta \sqrt{\frac{\cos^2 \left( \frac{t}{2} \theta \right) - \cos^2 \left( \frac{t}{2} \theta_0 \right)}{1 - \eta^2}} \tag{20}
\]

Now transform variables to
\[
\eta = \frac{\cos \left( \frac{t}{2} \theta \right) \eta}{\cos \left( \frac{t}{2} \theta_0 \right)} \tag{21}
\]
\[
\eta' \eta = -\frac{\sin \left( \frac{t}{2} \theta \right) d\theta}{2 \cos \left( \frac{t}{2} \theta_0 \right)} \tag{22}
\]

so
\[
T = -2 \sqrt{\frac{\alpha}{\varepsilon}} \int_{\theta_0}^{\theta} \eta' d\eta = 2 \sqrt{\frac{\alpha}{\varepsilon}} \left[ \sin^{-1} \eta \right]_{\theta_0}^{\theta} = 2 \sqrt{\frac{\alpha}{\varepsilon}} \frac{\theta - \theta_0}{\sqrt{1 - \eta^2}} \tag{23}
\]

and the amount of time is the same from any point, instead A.

References

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