A Study of Non-Linear Vibrations
Involving Contact with Friction

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Abstract
This article deals with the determination of the fundamental frequency of oscillation for a one-dimensional system subject to frictional contact non-linearities. A general solving method is given and some useful properties result from the analysis, such as the explicit non-linear dependency of the system response to the friction coefficient and contact clearance, allowing parametric and sensitivity analyses.

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1 Introduction
The following system, composed of a shaft of mass $m$ maintained in frictional contact in its housing under the effect of gravity $g$, is studied in this article. The shaft is apt to contact at its ends springs of stiffness $k_0$ (Figure 1). Let us denote as $J \geq 0$ the lateral gap in the symmetric position ($2J$ then represents the total clearance). Let $u(t)$ be the shaft lateral displacement measured with respect to the symmetric position and $u_{\text{max}}$ the maximum in amplitude of the lateral displacement.

The shaft and its housing are assumed to be rigid solids. Friction on shaft/housing contact interface is taken into account. Let us denote as $\mu \geq 0$ the associated friction coefficient. The aim of the study is to determine the influence of the lateral contact gap $J$ and friction coefficient $\mu$ on the natural frequency of oscillations of this system.
2 Assumptions and governing equations

A non-linear force-displacement model, typical of a system involving non-linear stiffness behaviour, is considered (Figure 2).

Let us thus introduce the following non-linear function $h$ of the lateral displacement, representing the force applied by the shaft on the springs:

$$h(u) = \begin{cases} k_0 (u + J) & \text{if } -u_{\text{max}} \leq u \leq -J \\ 0 & \text{if } -J \leq u \leq J \\ k_0 (u - J) & \text{if } J \leq u \leq u_{\text{max}} \end{cases}$$

(1)

where it has been assumed that $u_{\text{max}} \geq J$.

Free oscillations of the system are considered, i.e. in the absence of applied external forces. The differential equation governing the shaft dynamical motion then reads:
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\[ m \frac{d^2u}{dt^2} + h(u) = -\mu mg \text{ sgn}(\dot{u}) \]  \tag{2}

where \( \text{sgn} \) denotes the sign function, i.e.

\[ \text{sgn}(u) = \begin{cases} 
-1 & \text{if } \dot{u} < 0 \\
0 & \text{if } \dot{u} = 0 \\
1 & \text{if } \dot{u} > 0 
\end{cases} \]

3 Mathematical solving

Let us determine analytically the period \( T \) of oscillations of the system. Numerous numerical methods (e.g. \cite{1}, \cite{2}, \cite{3}, \cite{4}, \cite{6} and \cite{9}) can be found in the literature to solve equation (2) and determine the non-linear normal modes and frequencies.

Since we are principally interested in the determination of the first normal frequency of the system, a direct analytical method is preferred here.

The solving method consists first in writing

\[ \frac{d^2u}{dt^2} = \frac{1}{2} \frac{d(\dot{u})^2}{du} \cdot \]

Equation (2) can be then rewritten as:

\[ \frac{1}{2} m \dot{u} \left( \frac{du}{dt} \right)^2 = \left[ -h(u) - \mu mg \text{ sgn}(\dot{u}) \right] du \]  \tag{3}

That is:

\[ \left( \frac{du}{dt} \right)^2 = -\frac{2}{m} \int h(u) du - 2\mu g \text{ sgn}(\dot{u}) u + C \]  \tag{4}

where \( C \) is a constant to be determined.

On the displacement path \([-u_{\text{max}}, u_{\text{max}}]\), corresponding to the half-period \( T/2 \), one has \( \dot{u} > 0 \) and \( u = 0 \) for \( u = u_{\text{max}} \) at \( t = T/2 \). On the other hand, since \( \dot{u} > 0 \), \( \text{sgn}(\dot{u}) = 1 \).

Let \( H(u) = \int h(u) du \).

For \( u = u_{\text{max}} \), equation (4) reads \( 0 = -2H(u_{\text{max}})/m - 2\mu g \text{ sgn}(\dot{u}) u_{\text{max}} + C \), that is

\[ C = 2H(u_{\text{max}})/m. \]

It follows that:
\[ \left( \frac{du}{dt} \right)^2 = \frac{2}{m} \left[ H(u_{\text{max}}) - H(u) \right] - 2 \mu g \operatorname{sgn}(u) u \quad (5) \]

That is, since \( \dot{u} > 0 \):

\[ \frac{du}{dt} = \sqrt{2 \left[ \frac{1}{m} \int_{u_{\text{max}}}^{u} h(\xi) d\xi - \mu g u \right]} \quad (6) \]

Or equivalently:

\[ dt = \frac{du}{\sqrt{2 \left[ \frac{1}{m} \int_{u_{\text{max}}}^{u} h(\xi) d\xi - \mu g u \right]}} \quad (7) \]

By integrating (7) between \( u_{\text{max}} \) and \( u_{\text{max}} \), on the one hand and between \( t = 0 \) and \( t = T/2 \), on the other hand, one obtains:

\[ \frac{T}{2} = \int_{u_{\text{max}}}^{u_{\text{max}}} \frac{du}{\sqrt{2 \left[ \frac{1}{m} \int_{u_{\text{max}}}^{u} h(\xi) d\xi - \mu g u \right]}} \quad (8) \]

Finally, the period \( T \) is given as:

\[ T = \sqrt{2m} \int_{u_{\text{max}}}^{u_{\text{max}}} \frac{du}{\sqrt{2 \left[ \frac{1}{m} \int_{u_{\text{max}}}^{u} h(\xi) d\xi - \mu g u \right]}} \quad (9) \]

Let us perform the following decomposition, taking (1) into account:

\[ T = \sqrt{2m} \left[ \int_{u_{\text{max}}}^{u_{\text{max}}} \frac{du}{\sqrt{\int_{u_{\text{max}}}^{u_{\text{max}}} k_0(\xi + J) d\xi + \int_{J}^{u_{\text{max}}} k_0(\xi - J) d\xi - \mu g u}} \right. \]

\[ + \int_{J}^{u_{\text{max}}} \frac{du}{\sqrt{\int_{J}^{u_{\text{max}}} k_0(\xi - J) d\xi - \mu g u}} + \int_{u_{\text{max}}}^{u_{\text{max}}} \frac{du}{\sqrt{\int_{u_{\text{max}}}^{u_{\text{max}}} k_0(\xi - J) d\xi - \mu g u}} \]

That is:

\[ T = \sqrt{2m} \left[ \int_{u_{\text{max}}}^{u_{\text{max}}} \frac{du}{\sqrt{\int_{u_{\text{max}}}^{u_{\text{max}}} k_0(u_{\text{max}} - J) \left( u_{\text{max}} - 2J \right) - (Jk_0 + \mu mg)u - k_0 u^2}} \right. \]

\[ + \int_{J}^{u_{\text{max}}} \frac{du}{\sqrt{\int_{J}^{u_{\text{max}}} k_0(u_{\text{max}} - J)^2 - \mu g u}} + \int_{j}^{u_{\text{max}}} \frac{du}{\sqrt{\int_{j}^{u_{\text{max}}} (Jk_0 + \mu mg)u - k_0 u^2}} \]

Let us recall the following mathematical integration results:
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\[ \int \frac{dx}{\sqrt{c - bx - ax^2}} = \frac{1}{\sqrt{a}} \arcsin \left( \frac{a}{c + \frac{b^2}{4a}} \left( x + \frac{b}{2a} \right) \right) + \text{cte} \]

\[ \forall a, b, c > 0 \]

\[ \int \frac{dx}{\sqrt{a - bx}} = -\frac{2}{b} \sqrt{a - bx} + \text{cte} \]

\[ \int \frac{dx}{\sqrt{c + bx - ax^2}} = \frac{1}{\sqrt{a}} \arcsin \left( \frac{a}{c + \frac{b^2}{4a}} \left( x - \frac{b}{2a} \right) \right) + \text{cte} \]

It follows then that:

\[ I_1 = \int_{-\mu_{\text{max}}}^{\mu_{\text{max}}} \frac{du}{\sqrt{\frac{k_0}{2} (\mu_{\text{max}} - 2J) - (Jk_0 + \mu mg)u - \frac{k_0}{2} u^2}} \]

\[ = \frac{1}{\sqrt{\frac{k_0}{2}}} \left[ \arcsin \left( \frac{\mu mg}{k_0} \sqrt{\frac{k_0}{2} u_{\text{max}} (u_{\text{max}} - 2J) + \frac{(Jk_0 + \mu mg)^2}{2k_0}} \right) \right] \left[ -\arcsin \left( J - u_{\text{max}} + \frac{\mu mg}{k_0} \right) \sqrt{\frac{k_0}{2} u_{\text{max}} (u_{\text{max}} - 2J) + \frac{(Jk_0 + \mu mg)^2}{2k_0}} \right] \]

\[ I_2 = \int_{-\mu_{\text{max}}}^{\mu_{\text{max}}} \frac{du}{\sqrt{\frac{k_0}{2} (\mu_{\text{max}} - J)^2 - \mu mg u}} \]

\[ = -\frac{2}{\mu mg} \left[ \frac{k_0}{2} (u_{\text{max}} - J)^2 - \mu mg J - \frac{k_0}{2} (u_{\text{max}} - J)^2 + \mu mg J \right] \]
The exact expression of the period of oscillation as a function of friction coefficient \( \mu \), contact gap \( J \) and amplitude \( u_{\text{max}} \) therefore results in:

\[
T(\mu, J, u_{\text{max}}) = \sqrt{2m \left( I_1(\mu, J, u_{\text{max}}) + I_2(\mu, J, u_{\text{max}}) + I_3(\mu, J, u_{\text{max}}) \right)}
\]

(14)

The inverse of this quantity gives the natural frequency of oscillation \( f \) of the system.

4 Asymptotic study

4.1 Frictionless contact case

Let us consider the case where friction is negligible on the shaft/housing contact interface, i.e. \( \mu \ll 1 \). From equation (13), the passage to the limit \( \mu \to 0 \), in \( (I_1, I_2, I_3) \) leads to:

For integral \( I_1 \):

\[
\lim_{\mu \to 0} I_1 = -\frac{1}{\sqrt{2k_0}} \arcsin \left( \frac{k_0}{2} \right) \left[ \left( J - u_{\text{max}} \right) \sqrt{\frac{k_0}{2} u_{\text{max}} (u_{\text{max}} - 2J) + \frac{J^2 k_0}{2}} \right]
\]

(15)
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For $I_2$:

$$\lim_{\mu \to 0} I_2 = \frac{2J}{\sqrt{\frac{k_0}{2}(u_{\text{max}} - J)^2}} = \frac{2J}{\sqrt{\frac{k_0}{2}(u_{\text{max}} - J)}}$$

(16)

And for $I_3$:

$$\lim_{\mu \to 0} I_3 = \frac{1}{\sqrt{\frac{k_0}{2}}} \arcsin \left( \frac{u_{\text{max}} - J}{\sqrt{\frac{k_0}{2}u_{\text{max}}(u_{\text{max}} - 2J) + \frac{J^2k_0}{2}}} \right)$$

$$= \frac{1}{\sqrt{\frac{k_0}{2}}} \arcsin \left( \frac{u_{\text{max}} - J}{\sqrt{\frac{k_0}{2}(u_{\text{max}} - J)^2}} \right) = \frac{1}{\sqrt{\frac{k_0}{2}}} \arcsin(1) = \frac{\pi}{2} \frac{1}{\sqrt{\frac{k_0}{2}}}$$

(17)

Finally, the limit of the period is:

$$\lim_{\mu \to 0} T = \lim_{\mu \to 0} \sqrt{2m(I_1 + I_2 + I_3)} = 2\pi \sqrt{\frac{m}{k_0}} \left( 1 + \frac{2J}{\pi u_{\text{max}} - J} \right)$$

(18)

In the frictionless contact case, the expression of the natural frequency of the system is thus given as:

$$f = \frac{1}{2\pi} \sqrt{\frac{m}{k_0}} \left( 1 + \frac{2J}{\pi u_{\text{max}} - J} \right)$$

(19)

It is worth noting the non-linear dependency of the frequency in contact gap. The frequency also depends on the displacement amplitude, which is a property inherent to the response of solids in non-linear dynamics.

4.2 Frictionless contact – zero gap case

In this case, the passage to the limit $J \to 0$ in (18) leads to:

$$f = \frac{1}{2\pi} \sqrt{\frac{k_0}{m}}$$

(20)

One finds the classical well-known expression of the natural frequency for a mass-spring system, corresponding to the purely linear asymptotic behaviour of the considered non-linear system.
5 Sensitivity analysis example

Let us consider for instance the frictionless contact case (equation (19)). The numerical values used here are the following: \( m = 500 \text{ g} \), \( k_0 = 510^6 \text{ N.m}^{-1} \), \( u_{\text{max}} = 1 \text{ mm} \)

\[ \text{Hz} \]

\[ f \]

\[ 0 \leq J \leq u_{\text{max}}. \]

It is interesting to note in Figure 3 the non-linear increase in natural frequency with the decrease in lateral contact gap, showing the stiffening resulting from unilateral contact, as intuitively expected and numerically demonstrated [4].

6 Conclusion

The analytical approach adopted here has enabled to explicit the natural frequency in a case of non-linear free oscillations. In spite of the one-dimensional character of the considered mechanical system, the presented method could be generalized to systems involving multiple degrees of freedom and applied in the same spirit to similar bi-dimensional non-linear problems, such that the axisymmetric frictional contact model of a cylindrical shaft in a bearing housing. This can be useful in preliminary design analyses, not only to limit sounds and vibrations but also to prevent potential fatigue contact issues.

References

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