Solution of Flow in the Region Blocked Between Two Concentric Cylinders Normal to the Surface by Using Kelvin Transformation for Steady, Horizontal, Planar Flow in a Layer Having Uniform Thickness

M. Haluk Chelik and Avyt Asanov

Department of Mathematics, Kyrgyz-Turkish University Manas, Bishkek, Kyrgyzstan
yergok9999@gmail.com
avyt.asanov@yahoo.com

Inan Cinar

Department of Mathematics
Yuzuncu Yil University, Van, Turkey
inancinar@mynet.com

Abstract

This paper presents a mathematical approach and a solution technique to flow in the region blocked between two concentric cylinders normal to the surface of horizontal, planar, steady flow in a layer having uniform thickness by using Kelvin Transformation. Sequentially, a problem structure is formed and the problem is solved by using Kelvin Mapping Mathematics and the Fourier method. Considering the equations given in this paper, the flow and discharge at an arbitrary point in region Ω can be calculated with the aid of point (ξ, η) falling into Ω, region which is on a perpendicular section passing through that arbitrary point.

Keywords: Blocked region flow, concentric cylinders, Kelvin Mapping, Fourier method, elliptical differential equations, solution

Introduction: Solution of flow in the obstructed region between two concentric cylinders is a classical problem of fluid mechanics.
Dennis and Kocabiyik in their study, A general method of solving Oseen’s linearized equations for two-dimensional steady flow of a viscous incompressible fluid past a cylinder in an unbounded field is developed. The determination of the vorticity can be effected using conditions of an integral character deduced from the no-slip condition at the cylinder surface together with the conditions at large distances [1].

Frederic Dias and etal; Nonlinear waves in a forced channel flow of two contiguous homogeneous fluids of different densities are considered. Each fluid layer is of finite depth. The forcing is due to an obstruction lying on the bottom. The study is restricted to steady flows. First a weakly nonlinear analysis is performed. At leading order the problem reduces to a forced Korteweg-de Vries equation, except near a critical value of the ratio of layer depths which leads to the vanishing of nonlinear term. The weakly nonlinear results obtained by integrating the forced Korteweg-de Vries equation are validated by comparison with numerical results obtained by solving the full governing equations [2].

Rodrigo Surmas and etal presented a two-dimensional numerical simulation of fluid flow around a couple of identical circular cylinders aligned, respectively, along and orthogonal to the main-flow direction, at several distances A lattice-Boltzmann method is used. The forces resulting from fluid-solid interaction are calculated by considering the momentum exchanged between the fluid and the solid surfaces [3].

Steinmoeller Derek first considered the problem of prograde flow past a cylindrical obstacle on the β-plane. The problem is governed by the barotropic vorticity equation and is solved using a numerical method that is a combination of a finite difference method and a spectral method [4].

1. The Structure and Solution to the Problem

In this paper, solutions to the boundary-value problems for elliptic equations in the region Ω were constructed using Kelvin Mapping and methods described in [5, 6]. In particular it is proved that Kelvin Mapping is conformal. Problems similar to this paper is described in [7].

1.1 Definition of the problem: We examine two concentric cylinders with radii $R_1$ and $R$ satisfying $0 < R_1 < R$ and we take the center of the cylinders as the origin.

We denote the region between two cylinders by $\Omega$ and we assume that flow in this region is perpendicular to the axis of the cylinders, i.e. $\Omega = \left\{ (x, y) \in \mathbb{R}^2 : R_1 < \sqrt{x^2 + y^2} < R \right\}$. This region outside the cylinder with radius $R$ is denoted by $\Omega_1 = \left\{ (x, y) \in \mathbb{R}^2 : R < \sqrt{x^2 + y^2} < R_2 \right\}$, $R_2 = \frac{R^2}{R_1}$.

The aim is to calculate the flow and discharge at a point $X = (x, y)$ in a region $\Omega$ which falls into the cross-section perpendicular to cylinder axis with the help of another point $X' = (\xi, \eta)$ in a region of $\Omega_1$ which also falls into this
Our aim is to solve the equation

\[ u_{xx} + u_{yy} + (x^3 + xy^2)\alpha(\sqrt{x^2 + y^2})u_x + \\
+ (x^2y + y^3)\alpha(\sqrt{x^2 + y^2})u_y + \\
+ \beta(\sqrt{x^2 + y^2})u = f(x, y), (x, y) \in \Omega, \tag{1} \]

with the next boundary conditions

\[ u \bigg|_{\sqrt{x^2+y^2}=R_1} = f_1(\theta), -\pi \leq \theta \leq \pi, \tag{2} \]

\[ u \bigg|_{\sqrt{x^2+y^2}=R} = f_2(\theta), -\pi \leq \theta \leq \pi, \tag{3} \]

where \(0 < R_1 < R\), \(\Omega = \{(x, y) \in R^2 : R_1 < \sqrt{x^2+y^2} < R\}\). \(\alpha(t)\) and \(\beta(t)\) are functions known from \(C[R_1; R]\), \(f(x, y)\) is a function known from \(C[\Omega]\), \(f_i(\theta)\) is function known from \(C^1[-\pi; \pi]\), \(f_i(-\pi) = f_i(\pi), i = 1, 2\).

1.2 Solution of the problem: With the aim stated in the Section 1.1., we construct a transformation which provides a one to one-to-one correspondence between a point in \(\Omega\) and a point in \(\Omega_1\).

This transformation must move point \(X\) to point \(X'\) or it must do the reverse. Then, we define the rule for this operation as:

\[ |OX||OX'| = R^2. \tag{4} \]

Since points \(O(0, 0), X(x, y)\) and \(X' = (\xi, \eta)\) are on the same line, if \(\lambda \in R, \ OX' = OX + \lambda OX\) can be stated, then \((\xi, \eta) = (x + \lambda x, y + \lambda y)\) can be obtained. By using the equality of two vectors, we can obtain:

\[ \xi = (1 + \lambda)x, \]
\[ \eta = (1 + \lambda)y. \tag{5} \]

If the squares of both sides of Equation (5) are taken, we obtain:

\[ \xi^2 = (1 + \lambda)^2x^2, \]
\[ \eta^2 = (1 + \lambda)^2y^2. \tag{6} \]

If we substitute the values in Equation (6) for Equation (4), we obtain:

\[ |OX||OX'| = \sqrt{x^2+y^2}\sqrt{\xi^2+\eta^2} = \sqrt{x^2+y^2}\sqrt{(1+\lambda)^2(x^2+y^2)} = \\
= (1+\lambda)(x^2+y^2) = R^2. \]
Then:

\[ 1 + \lambda = \frac{R^2}{x^2 + y^2}. \]  

(7)

If we substitute Equation (7) for Equation (5), we get:

\[ \xi = \frac{R^2}{x^2 + y^2} x, \]

\[ \eta = \frac{R^2}{x^2 + y^2} y. \]  

(8)

If \( X \) and \( X' \) are interchanged, we obtain:

\[ x = \frac{R^2}{\xi^2 + \eta^2} \xi, \]

\[ y = \frac{R^2}{\xi^2 + \eta^2} \eta. \]  

(9)

The equation (9) can be expressed as:

\[ x = \frac{R^2}{|X'|^2} \xi, \]

\[ y = \frac{R^2}{|X'|^2} \eta. \]  

(10)

From (8) it can be shown by elementary operation that:

\[ \xi_x = R^2 \frac{y^2 - x^2}{(x^2 + y^2)^2}, \quad \xi_y = -2R^2 \frac{xy}{(x^2 + y^2)^2}, \]

\[ \eta_x = -2R^2 \frac{xy}{(x^2 + y^2)^2}, \quad \eta_y = R^2 \frac{x^2 - y^2}{(x^2 + y^2)^2}, \]

\[ \xi_{xx} = \frac{2(x^3 - 3xy^2)}{(x^2 + y^2)^3} R^2, \quad \xi_{xy} = \frac{2(3x^2y - y^3)}{(x^2 + y^2)^3} R^2, \]

\[ \xi_{yy} = \frac{2(-x^3 + 3xy^2)}{(x^2 + y^2)^3} R^2, \quad \eta_{xx} = \frac{2(3x^2y - y^3)}{(x^2 + y^2)^3} R^2, \]

\[ \eta_{xy} = \frac{2(-x^3 + 3xy^2)}{(x^2 + y^2)^3} R^2, \quad \eta_{yy} = \frac{2(y^3 - 3x^2y)}{(x^2 + y^2)^3} R^2. \]

Then

\[ u_x = \frac{R^2}{(x^2 + y^2)^2} [(y^2 - x^2)u_\xi - 2xyu_\eta], \]

\[ u_y = \frac{R^2}{(x^2 + y^2)^2} [(-2xy)u_\xi + (x^2 - y^2)u_\eta], \]
From (14) we have

\[
\begin{align*}
\frac{R^4}{(x^2 + y^2)^2}[(y^2 - x^2)^2 u_{\xi \xi} + 4xy(x^2 - y^2)u_{\xi \eta} + 4x^2y^2 u_{\eta \eta}] + \\
+ \frac{2R^2}{(x^2 + y^2)^3}[(x^3 - 3xy^2)u_{\xi} + (3x^2y - y^3)u_{\eta}],
\end{align*}
\]

\[
\begin{align*}
\frac{R^4}{(x^2 + y^2)^2}[4x^2y^2 u_{\xi \xi} - 4xy(x^2 - y^2)u_{\xi \eta} + (x^2 - y^2)^2 u_{\eta \eta}] + \\
+ \frac{2R^2}{(x^2 + y^2)^3}[-x^3 + 3xy^2)u_{\xi} + (y^3 - 3x^2 y)u_{\eta}],
\end{align*}
\]

\[
\begin{align*}
u_{xx} + u_{yy} = \frac{R^4}{(x^2 + y^2)^2}(u_{\xi \xi} + u_{\eta \eta}),
\end{align*}
\]

(11)

\[
\begin{align*}
[(x^2 - y^2)x + 2xy] \alpha(\sqrt{x^2 + y^2})u_x + [2x^2 y + (y^2 - x^2)y] \alpha(\sqrt{x^2 + y^2})u_y = \\
\alpha(\sqrt{x^2 + y^2}) \frac{R^2}{(x^2 + y^2)^2} \left\{[(y^2 - x^2)u_{\xi} - 2xyu_{\eta}][(x^2 - y^2)x + 2xy^2] + \\
+ [2x^2 y + (y^2 - x^2)y](-2xy)u_{\xi} + (x^2 - y^2)u_{\eta}\right\} = \\
\frac{R^2 \alpha(\sqrt{x^2 + y^2})}{(x^2 + y^2)^2} \left\{[-(x^2 - y^2)^2x - 4x^3 y^2]u_{\xi} + \\
+ [-4x^2 y^3 - (x^2 - y^2)^2)w_{\eta}]\right\} = -R^2 \alpha(\sqrt{x^2 + y^2})(xu_{\xi} + yu_{\eta}).
\end{align*}
\]

Here

\[
\alpha(\sqrt{x^2 + y^2}) \left\{[(x^2 - y^2)x + 2xy^2]u_x + [2x^2 y + (y^2 - x^2)y]u_y\right\} = \\
= -\alpha(\frac{R^2}{\sqrt{x^2 + y^2}}) \frac{R^4}{\xi^2 + \eta^2}[\xi u_{\xi} + \eta u_{\eta}].
\]

(12)

By taking into account (11),(12) and (8) from (1) we obtain:

\[
\begin{align*}
\frac{\xi^2 + \eta^2}{R^2} (u_{\xi \xi} + u_{\eta \eta}) - \frac{R^4}{\xi^2 + \eta^2} \alpha(\frac{R^2}{\sqrt{\xi^2 + \eta^2}})(\xi u_{\xi} + \eta u_{\eta}) + \\
+ \beta(\frac{R^2}{\sqrt{\xi^2 + \eta^2}})u = f \left( \frac{R^2}{\xi^2 + \eta^2} \xi, \frac{R^2}{\xi^2 + \eta^2} \eta \right).
\end{align*}
\]

(13)

We proceed to the new transformations:

\[
\begin{align*}
\xi = r \cos \theta, \\
\eta = r \sin \theta, \\
-\pi \leq \theta \leq \pi.
\end{align*}
\]

(14)

From (14) we have

\[
\begin{align*}
\xi_r = \cos \theta, \xi_\theta = -r \sin \theta, \eta_r = \sin \theta, \eta_\theta = r \cos \theta,
\end{align*}
\]
\[ \xi_{rr} = 0, \quad \xi_{r\theta} = -\sin \theta, \quad \xi_{\theta\theta} = -r \cos \theta, \]
\[ \eta_{rr} = 0, \quad \eta_{r\theta} = \cos \theta, \quad \eta_{\theta\theta} = -r \sin \theta. \]

Then:
\[
\begin{cases} 
    u_r = u_\xi \cos \theta + u_\eta \sin \theta, \\
    u_\theta = u_\xi (-r \sin \theta) + u_\eta r \cos \theta. 
\end{cases} \tag{15}
\]

From (15) we obtain:
\[
\begin{cases} 
    u_\xi = \cos \theta u_r - \frac{1}{r} \sin \theta u_\theta, \\
    u_\eta = \sin \theta u_r + \frac{1}{r} \cos \theta u_\theta, 
\end{cases} \tag{16}
\]

\[ u_{rr} = \cos^2 \theta u_{\xi\xi} + 2 \cos \theta \sin \theta u_{\xi\eta} + \sin^2 \theta u_{\eta\eta}, \tag{17} \]
\[ \frac{1}{r^2} u_{\theta\theta} = \sin^2 \theta u_{\xi\xi} - 2 \cos \theta \sin \theta u_{\xi\eta} + \cos^2 \theta u_{\eta\eta} - \frac{1}{r} \cos \theta u_\xi - \frac{1}{r} \sin \theta u_\eta. \tag{18} \]

Taking into account (15), from (17) and (18) we obtain:
\[ u_{\xi\xi} + u_{\eta\eta} = u_{rr} + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r} u_r \tag{19} \]

Then taking into account (14) and (16) we have:
\[ \xi u_\xi + \eta u_\eta = ru_r. \tag{20} \]

On the strength of (14), (19) and (20), equation (13) assumes the next form:
\[
\begin{align*}
    u_{rr} + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r} u_r - \frac{R^8}{r^5} & \alpha \left( \frac{R^2}{r} \right) u_r + \frac{R^4}{r^4} \beta \left( \frac{R^2}{r} \right) u = \\
    &= R^4 \int \frac{R^2}{r} \cos \theta, \quad \frac{R^2}{r} \sin \theta, \tag{21}
\end{align*}
\]

where \((r, \theta) \in [R; R_2] \times [-\pi; \pi]\). From conditions (2) and (3) we obtain:
\[ u|_{r=R} = f_2(\theta), \quad -\pi \leq \theta \leq \pi, \tag{22} \]
\[ u|_{r=R_2} = f_1(\theta), \quad -\pi \leq \theta \leq \pi. \tag{23} \]

We use the method of Fourier-series expansion to solve the problem (21)-(23). At first we expand known functions \( R^4 \int \frac{R^2}{r} \cos \theta, \frac{R^2}{r} \sin \theta \), \( f_1(\theta) \) and \( f_2(\theta) \) by \( \theta \) at the interval \([-\pi, \pi]\) in the Fourier series.
\[
\frac{R^4}{r^4} \int \frac{R^2}{r} \cos \theta, \frac{R^2}{r} \sin \theta = \frac{1}{2} a_0 + \sum_{k=1}^{\infty} [a_k(r) \cos k \theta + b_k(r) \sin k \theta], \tag{24}
\]
where:

\[
\begin{align*}
  a_k(r) &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{R^4}{r} f \left( \frac{R^2}{r} \cos \theta, \frac{R^2}{r} \sin \theta \right) \cos k \theta d\theta, \quad (k = 0, 1, 2, \ldots), \\
  b_k(r) &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{R^4}{r} f \left( \frac{R^2}{r} \cos \theta, \frac{R^2}{r} \sin \theta \right) \sin k \theta d\theta, \quad (k = 1, 2, \ldots), \\
  p_{ik} &= \frac{1}{\pi} \int_{-\pi}^{\pi} f_i(\theta) \cos k \theta d\theta, \quad (i = 1, 2; \quad k = 0, 1, 2, \ldots), \\
  q_{ik} &= \frac{1}{\pi} \int_{-\pi}^{\pi} f_i(\theta) \sin k \theta d\theta, \quad (i = 1, 2; \quad k = 1, 2, \ldots).
\end{align*}
\]

We are seeking a solution to the problem (21)-(23) in the next form:

\[
u(r, \theta) = \frac{1}{2} u_0(r) + \sum_{k=1}^{\infty} [u_k(r) \cos k \theta + v_k(r) \sin k \theta],
\]

where \(u_0(r), u_k(r)\) and \(v_k(r)\) are unknown functions. By substituting (27) for (21) and taking into account (24) we obtain:

\[
\begin{align*}
  &\frac{1}{2} \left[ u''_0(r) + \frac{1}{r} u'_0(r) - \frac{R^8}{r^5} \alpha \left( \frac{R^2}{r} \right) u'_0(r) + \frac{R^4}{r^4} \beta \left( \frac{R^2}{r} \right) u_0(r) \right] + \\
  &+ \sum_{k=1}^{\infty} \left\{ u''_k(r) + \frac{1}{r} u'_k(r) - \frac{R^8}{r^5} \alpha \left( \frac{R^2}{r} \right) u'_k(r) + \frac{R^4}{r^4} \beta \left( \frac{R^2}{r} \right) u_k(r) - \frac{k^2}{r^2} u_k(r) \right\} \cos k \theta + \\
  &+ \left[ v''_k(r) + \frac{1}{r} v'_k(r) - \frac{R^8}{r^5} \alpha \left( \frac{R^2}{r} \right) v'_k(r) + \frac{R^4}{r^4} \beta \left( \frac{R^2}{r} \right) v_k(r) - \frac{k^2}{r^2} v_k(r) \right] \sin k \theta \\
  &= \frac{1}{2} u_0(r) + \sum_{k=1}^{\infty} [a_k(r) \cos k \theta + b_k(r) \sin k \theta].
\end{align*}
\]

From this, we obtain the following system of equations for definition \(u_0(r), u_k(r), v_k(r)(k = 1, 2, \ldots)\):

\[
\begin{align*}
  &u''_0(r) + \frac{1}{r} u'_0(r) - \frac{R^8}{r^5} \alpha \left( \frac{R^2}{r} \right) u'_0(r) + \frac{R^4}{r^4} \beta \left( \frac{R^2}{r} \right) u_0(r) = a_0(r), \\
  &u''_k(r) + \frac{1}{r} u'_k(r) - \frac{R^8}{r^5} \alpha \left( \frac{R^2}{r} \right) u'_k(r) + \frac{R^4}{r^4} \beta \left( \frac{R^2}{r} \right) u_k(r) - \frac{k^2}{r^2} u_k(r) = a_k(r), \\
  &v''_k(r) + \frac{1}{r} v'_k(r) - \frac{R^8}{r^5} \alpha \left( \frac{R^2}{r} \right) v'_k(r) + \frac{R^4}{r^4} \beta \left( \frac{R^2}{r} \right) v_k(r) - \frac{k^2}{r^2} v_k(r) = b_k(r), \quad (k = 1, 2, \ldots).
\end{align*}
\]

On the strength of (25) and in taking (27) from (22) and (23) we have

\[
\begin{align*}
  &u_0(R) = p_{20}, \quad u_0(R_2) = p_{10}, \\
  &u_k(R) = p_{2k}, \quad u_k(R_2) = p_{1k}, \\
  &v_k(R) = q_{2k}, \quad v_k(R_2) = q_{1k}, \quad (k = 1, 2, \ldots).
\end{align*}
\]
By solving the system of equations (28) with the boundary condition (29) we uniquely define unknown functions $u_0(r), u_k(r), v_k(r)(k = 1, 2, \ldots)$. Thus by the formula (27) we define the required function $u(r, \theta)$.

Imposing conditions $f(x, y), f_1(\theta)$ and $f_2(\theta)$ we can investigate the uniform convergence of the series (27).

**Example 1.** Consider boundary-value problem (1)-(3) in conditions when $\alpha(t) = 0$ and $\beta(t) = -\frac{c}{a}, c > 0$ for all $t \in [R_1, R]$.

We can input the designations:
\[
S_k = \sqrt{k^2 + c}, \quad (k = 0, 1, 2, \ldots). \tag{30}
\]

Then by the method of variation of arbitrary constants it can be shown that the systems (28) have the following solutions
\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{s_k}{2S_k} \int_R^r a_k(r)r^{-S_k+1}dr - \frac{\xi_k}{2S_k} \int_R^r a_k(r)r^{S_k+1}dr + \\
+ c_{k1}R^{S_k} + c_{k2}R^{-S_k}, \quad r \in [R, R_2], \quad (k = 0, 1, 2, \ldots),
\end{array} \right.
\end{align*}
\]
\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{s_k}{2S_k} \int_R^r b_k(r)r^{-S_k+1}dr - \frac{\xi_k}{2S_k} \int_R^r b_k(r)r^{S_k+1}dr + \\
+ c_{k1}R^{S_k} + c_{k2}R^{-S_k}, \quad r \in [R, R_2], \quad (k = 1, 2, 3, \ldots)
\end{array} \right.
\end{align*}
\] \tag{31}

where $S_k$ are specified by formula (30), and $c_{01}, c_{02}, c_{k1}, c_{k2}, c_{n1}^1, c_{n2}^1$ are arbitrary constants.

Taking into account system (31) from (29) we obtain the next system for defining unknown constants $c_{01}, c_{02}, c_{k1}, c_{k2}, c_{n1}^1, c_{n2}^1$.
\[
\begin{align*}
\left\{ \begin{array}{l}
c_{k1}R^{S_k} + c_{k2}R^{-S_k} = p_{2k}, \\
c_{k1}R^{S_k} + c_{k2}R^{-S_k} = p_{1k}, \quad (k = 0, 1, 2, \ldots),
\end{array} \right.
\end{align*}
\]
\[
\begin{align*}
\left\{ \begin{array}{l}
c_{n1}R^{S_n} + c_{n2}R^{-S_n} = q_{2n}, \\
c_{n1}R^{S_n} + c_{n2}R^{-S_n} = q_{1n}, \quad (n = 1, 2, 3, \ldots)
\end{array} \right.
\] \tag{32}

where
\[
p_{1k} = p_k - \frac{R^{S_k}}{2S_k} \int_R^r a_k(r)r^{-S_k+1}dr + \frac{R^{S_k}}{2S_k} \int_R^r a_k(r)r^{S_k+1}dr,
\]
\[
q_{1n} = q_n - \frac{R^{S_n}}{2S_n} \int_R^r a_n(r)r^{-S_n+1}dr + \frac{R^{S_n}}{2S_n} \int_R^r a_n(r)r^{S_n+1}dr. \tag{33}
\]

By solving system (32) we define unknown constants $c_{01}, c_{02}, c_{k1}, c_{k2}, c_{n1}^1, c_{n2}^1$.
\[
\begin{align*}
\left\{ \begin{array}{l}
c_{k1} = \frac{p_{2k}R^{-S_k} - (p_{1k}R^{S_k} - p_{2k}R^{-S_k})R^{2S_k}}{R^{2S_k}R^{-2S_k}}, \\
c_{k2} = \frac{p_{1k}R^{S_k} - p_{2k}R^{-S_k}}{R^{2S_k}R^{-2S_k}}, \quad (k = 0, 1, 2, \ldots),
\end{array} \right.
\end{align*}
\]
\[
\begin{align*}
\left\{ \begin{array}{l}
c_{n1} = \frac{q_{2n}R^{-S_n} - (q_{1n}R^{S_n} - q_{2n}R^{-S_n})R^{2S_n}}{R^{2S_n}R^{-2S_n}}, \\
c_{n2} = \frac{q_{1n}R^{S_n} - q_{2n}R^{-S_n}}{R^{2S_n}R^{-2S_n}}, \quad (n = 1, 2, 3, \ldots).
\end{array} \right.
\] \tag{34}
Thus with the aid of the formulas (34), (31) and (27) we can define the unknown function \( u(r, \theta) \).

**Example 2:** Consider boundary-value problem (1)-(3) in conditions when \( \alpha(t) = 0, \beta(t) = -\frac{3}{2t} \) for \( t \in [R_1, R] \), \( f(x, y) = x + y = \sqrt{x^2 + y^2}(\cos \theta + \sin \theta) \) for \( (x, y) \in \Omega \), \( f_2(\theta) = \frac{x}{\sqrt{x^2 + y^2}} = \cos \theta \), \( f_1(\theta) = 0 \) for \( \theta \in [-\pi, \pi] \), \( R = 1 \), \( R_1 = \frac{1}{2}, \ R_2 = 2 \). Then, from (26) and (30) we obtain:

\[
\begin{align*}
\begin{cases}
a_0(r) = 0, \ a_1(r) = b_1(r) = r^{-5}, \ a_k(r) = b_k(r) = 0, \ r \in [1, 2], \\
p_{21} = 1, \ p_{11} = q_{11} = q_{21} = 0, \ p_{ik} = q_{ik} = 0, \ i = 1, 2, \\
S_0 = \sqrt{2}, \ S_1 = 2, \ S_k = \sqrt{k^2 + 3}, \ k \in \mathbb{N}, \ k \geq 2.
\end{cases}
\end{align*}
\]

(35)

Taking into account (35), we can obtain the following from (33):

\[
p_{11}^1 = q_{11}^1 = -\frac{13}{80}, \ p_{20}^1 = q_{20}^1 = 0, \ p_{2k}^1 = q_{2k}^1 = 0, \ k \in \mathbb{N}, \ k \geq 2.
\]

(36)

On the strength (35) and (36), from (34) we have:

\[
\begin{align*}
\begin{cases}
c_{11} = -\frac{11}{100}, \ c_{12} = \frac{111}{100}, \ c_{11}^1 = -\frac{13}{300}, \ c_{12}^1 = \frac{13}{300}, \ c_{01} = c_{02} = 0, \\
c_{k1} = c_{k2} = c_{k1}^1 = c_{k2}^1 = 0, \ k \in \mathbb{N}, \ k \geq 2.
\end{cases}
\end{align*}
\]

(37)

Taking into account (35) and (37), from (31) we obtain:

\[
\begin{align*}
\begin{cases}
u_1(r) = \frac{1}{5} r^{-3} - \frac{3}{50} r^2 + \frac{43}{50} r^{-2}, \\
u_1(r) = \frac{1}{5} r^{-3} + \frac{13}{150} r^2 - \frac{31}{150} r^{-2}, \ r = (x^2 + y^2)^{-\frac{1}{2}}, \\
u_0(r) = u_k(r) = v_k(r) = 0, \ t \in [1, 2], \ k \in \mathbb{N}, \ k \geq 2.
\end{cases}
\end{align*}
\]

(38)

On the strength (38), from (27) we have:

\[
\begin{align*}
u(x, y) = u_1(r)\frac{x}{\sqrt{x^2 + y^2}} + v_1(r)\frac{y}{\sqrt{x^2 + y^2}} = \\
= \left( \frac{1}{5} r^{-3} + \frac{1}{150} r^2 - \frac{31}{150} r^{-2} \right) \frac{x + y}{\sqrt{x^2 + y^2}} + \left( -\frac{1}{15} r^2 + \frac{16}{15} r^{-2} \right) \frac{x}{\sqrt{x^2 + y^2}}.
\end{align*}
\]

From here, we obtain

\[
u(x, y) = \frac{1}{5} (x + y) [x^2 + y^2 + \frac{1}{30} (x^2 + y^2)^{-\frac{1}{2}} - \frac{31}{30} (x^2 + y^2)^{\frac{1}{2}}] + \\
x [-\frac{1}{15} (x^2 + y^2)^{-\frac{1}{2}} + \frac{16}{15} (x^2 + y^2)^{\frac{1}{2}}], \ (x, y) \in \Omega.
\]

(39)
For $\sqrt{x^2 + y^2} = R_1 = \frac{1}{2}$ and $\sqrt{x^2 + y^2} = 1$, from (39), we have

$$u(x, y)|_{\sqrt{x^2 + y^2} = \frac{1}{2}} = \frac{1}{5}(x+y) \left( \frac{1}{4} + \frac{1}{30} \cdot 8 - \frac{31}{30} \cdot \frac{1}{2} \right) + x \left( -\frac{1}{15} \cdot 8 + \frac{16}{15} \cdot \frac{1}{2} \right) = 0,$$

$$u(x, y)|_{\sqrt{x^2 + y^2} = 1} = \frac{1}{5}(x+y) \left( 1 + \frac{1}{30} - \frac{31}{30} \right) + x \left( -\frac{1}{15} + \frac{16}{15} \right) = x = \cos \theta, \ \theta \in [-\pi, \pi].$$

Differentiating (39), we obtain:

$$u_x = \frac{1}{5} \left[ x^2 + y^2 + \frac{1}{30} \left( x^2 + y^2 \right)^{-\frac{3}{2}} - \frac{31}{30} \left( x^2 + y^2 \right)^{\frac{1}{2}} \right] +$$

$$+ \frac{1}{5} (x^2 + xy) \left[ 2 - \frac{1}{10} (x^2 + y^2)^{-\frac{3}{2}} - \frac{31}{30} (x^2 + y^2)^{-\frac{1}{2}} \right] - \frac{1}{15} (x^2 + y^2)^{-\frac{1}{2}} +$$

$$+ \frac{16}{15} (x^2 + y^2)^{\frac{1}{2}} + x^2 \left[ \frac{1}{5} (x^2 + y^2)^{-\frac{3}{2}} + \frac{16}{15} (x^2 + y^2)^{-\frac{1}{2}} \right],$$

$$u_{xx} = \frac{1}{5} (3x + y) \left[ 2 - \frac{1}{10} (x^2 + y^2)^{-\frac{3}{2}} - \frac{31}{30} (x^2 + y^2)^{-\frac{1}{2}} \right] +$$

$$+ \frac{1}{5} (x^3 + x^2 y) \left[ \frac{1}{2} (x^2 + y^2)^{-\frac{3}{2}} + \frac{31}{30} (x^2 + y^2)^{-\frac{1}{2}} \right] + \frac{3}{5} x \left[ (x^2 + y^2)^{-\frac{3}{2}} +$$

$$+ \frac{16}{3} (x^2 + y^2)^{-\frac{1}{2}} \right] + x^2 \left[ -(x^2 + y^2)^{-\frac{3}{2}} - \frac{16}{15} (x^2 + y^2)^{-\frac{1}{2}} \right], \quad (40)$$

$$u_y = \frac{1}{5} \left[ x^2 + y^2 + \frac{1}{30} (x^2 + y^2)^{-\frac{3}{2}} - \frac{31}{30} (x^2 + y^2)^{\frac{1}{2}} \right] +$$

$$+ \frac{1}{5} (xy + y^2) \left[ 2 - \frac{1}{10} (x^2 + y^2)^{-\frac{3}{2}} - \frac{31}{30} (x^2 + y^2)^{-\frac{1}{2}} \right] +$$

$$+ \frac{1}{5} xy \left[ (x^2 + y^2)^{-\frac{3}{2}} + \frac{16}{3} (x^2 + y^2)^{-\frac{1}{2}} \right],$$

$$u_{yy} = \frac{1}{5} (x + 3y) \left[ 2 - \frac{1}{10} (x^2 + y^2)^{-\frac{3}{2}} - \frac{31}{30} (x^2 + y^2)^{-\frac{1}{2}} \right] +$$

$$+ \frac{1}{5} (y^3 + xy^2) \left[ \frac{1}{2} (x^2 + y^2)^{-\frac{3}{2}} + \frac{31}{30} (x^2 + y^2)^{-\frac{1}{2}} \right] +$$

$$+ \frac{1}{5} x \left[ (x^2 + y^2)^{-\frac{3}{2}} + \frac{16}{3} (x^2 + y^2)^{-\frac{1}{2}} \right] +$$

$$+ y^2 x \left[ -(x^2 + y^2)^{-\frac{3}{2}} - \frac{16}{15} (x^2 + y^2)^{-\frac{1}{2}} \right]. \quad (41)$$

Then taking into account (39), (40) and (41) we have:

$$u_{xx} + u_{yy} - \frac{3}{x^2 + y^2} u = (x+y) \left( \frac{8}{5} - \frac{3}{5} \right) + (x+y) \left[ (x^2 + y^2)^{-\frac{3}{2}} + \frac{16}{5} \right] +$$

$$+ \frac{1}{5} \left( \frac{8}{5} - \frac{3}{5} \right) \left( x^2 + y^2 \right)^{-\frac{3}{2}} - \frac{4}{50} + \frac{1}{10} - \frac{1}{50}.$$
\[
+ (x^2 + y^2)^{-\frac{1}{2}} \left( -\frac{31}{30} \cdot \frac{4}{5} + \frac{1}{2} \cdot \frac{31}{30} + \frac{31}{30} \cdot \frac{3}{5} \right) + \\
+ x \left[ (x^2 + y^2)^{-\frac{1}{2}} \left( \frac{3}{5} + \frac{1}{2} + \frac{1}{5} \right) + (x^2 + y^2)^{-\frac{1}{2}} \left( \frac{16}{5} + \frac{16}{15} - \frac{48}{15} \right) \right] + \\
x(x^2 + y^2) \left[ -(x^2 + y^2)^{-\frac{1}{2}} - \frac{16}{15}(x^2 + y^2)^{-\frac{1}{2}} \right] = x + y, \quad (x, y) \in \Omega.
\]

If \( \alpha(t) = \beta(t) = 0 \) for \( t \in [R_1, R] \) and \( f(x, y) = 0 \) for \( (x, y) \in \Omega \), then from (13) we have:

\[
u_{\xi \xi} + u_{\eta \eta} = 0, \quad (\xi, \eta) \in \Omega_1. \tag{42}\]

From (10), we obtain:

\[
X = (x, y) = \frac{R^2}{|\xi, \eta|^2}(\xi, \eta) = \frac{R^2}{|X'|^2}X',
\]

\[
dX = R^2 \frac{|X'|^2 dX' - (2X'dX')X'}{|X'|^4} = R^2 \frac{-dX'}{|X'|^2}.
\]

From this

\[
|dX|^2 = R^4 \frac{|dX'|^2}{|X'|^4}.
\]

If we write

\[
\lambda(X') = \frac{R^4}{|X'|^4},
\]

then

\[
|dX|^2 = \lambda(X')|dX'|^2. \tag{43}\]

Thus, expression (42) and (43) convince us that the transformation we made with the help of Equation (10) is ”conformed”, and the direction of flow is preserved.

**Example 3:** When we calculate the flow and discharge at point \((x, y)\), the distance of point the region of \((\xi, \eta)\) in \(\Omega_1\) is related to \(R_1\) and \(R\). For example, if \(R_1 = 1\) and \(R = 5\), the point \((\xi, \eta)\) can be a maximum 25 units away from the origin.

**2. Conclusion**

Considering the equation (1) for \( \alpha(t) = \beta(t) = 0 \) at \( t \in [R_1, R] \) and \( f(x, y) = 0 \) at \((x, y) \in \Omega\) with the boundary conditions (2) and (3), we can calculate the flow and discharge at an arbitrary point in region \( \Omega \) with the help of point \((\xi, \eta)\) falling into region \( \Omega_1 \) which is on a perpendicular section passing through that arbitrary point.
References


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