Flow through a Membrane of Porous Cylindrical Particles with Varying Permeability

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Abstract

The present paper concerns the flow through random assemblage of porous cylindrical particles of radially varying permeability. Brinkman equation is used for formulation of flow through porous medium. Effect of various parameters on the permeability of swarm is being discussed graphically.

Mathematics Subject Classification: 76D07

Keywords: Brinkman equation, Modified Bessel functions, Drag force

Introduction

Flow through random assemblage of particles has been a topic of interest from last fifty years due to its applications in the membrane filtration process [6]. However, the mathematical formulation of such problems is a complex task as one has to analyze each and every particle in order to get information on flow field. To overcome this problem, cell model technique was introduced. This technique is used to replace a system of chaotically distributed particles in to a periodic array of particles confined in liquid cells. Uchida [11] gave this concept in which he singled out a particle from the swarm and assumed it to be confined within a cubic cell acting as a fluid envelope. A major drawback of the model is that of difference in outer and inner geometry.

In order to overcome the limitations of above model Happel [4, 5] assumed vanishng of shearing stress on the cell signifying no friction between the particles of swarms due to interaction. On the other hand Kuwabarara [8] assumed nil vorticity on the cell surface i.e. the flow is of potential kind. Both the formulations give almost same results but in Kuwabara’s case there is slight exchange of mechanical energy between cell and environment. Mehta-Morse

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In this paper, we have presented flow through random assemblage of porous cylindrical particles with variable permeability enclosing an impermeable core, and discussed the effect of various parameters on the membrane.

**Mathematical Formulation**

Here, we have considered an axi-symmetric Stokes flow of a viscous incompressible fluid through a swarm of porous cylindrical particles of radii \( \hat{b} \), each enclosing an impermeable core of radius \( \hat{a} \). This model is equivalent to a coaxial porous cylindrical shell enclosing a solid core (fig.-1). It is assumed that axes of all cylinders are parallel. A of cell model, each porous shell is assumed to be enveloped by a concentric cylinder of radius \( \hat{c} \) (\( \hat{c} > \hat{b} \)), named as cell surface. The Stokes flow of a Newtonian fluid with absolute fluid viscosity is assumed to be steady and axi-symmetric. We assume that the fluid is approaching towards the cell surface as well as partially passing through the composite cylinder along the axis of cylinders (z-axis) with velocity \( U \) from left to right. The radius \( \hat{c} \) of hypothetical cell is chosen in such a way that the particle volume fraction of the porous cylinder is equal to the particle volume fraction of the cell, i.e. relative to this composite cylinder (i.e. a core with porous shell) in the hypothetical cell, i.e.

\[
\gamma = \pi \hat{b}^2 / \pi \hat{c}^2
\]

(1)

From Stoke’s Equation in the region \( \hat{b} \) to \( \hat{c} \), Flow is axi-symmetric i. e. \( v_\theta = 0 \)

\[
\tilde{\mu}^{(1)} \frac{1}{\hat{r}} \frac{d}{d\hat{r}} \left( \hat{r} \frac{d\tilde{u}^{(1)}}{d\hat{r}} \right) = \frac{d\tilde{p}}{d\hat{z}}, \quad (\hat{b} < \hat{r} < \hat{c})
\]

(2)

Fully developed flow in porous region from \( b \) to \( a \), Brink Man’s equation gives:

\[
\tilde{\mu}^{(2)} \frac{1}{\hat{r}} \frac{d}{d\hat{r}} \left( \hat{r} \frac{d\tilde{u}^{(2)}}{d\hat{r}} \right) - \frac{\tilde{\mu}^{(1)}}{k} \tilde{u}^2 = \frac{d\tilde{p}}{d\hat{z}}, \quad (\hat{a} < \hat{r} < \hat{b})
\]

(3)

Where the tilde denotes dimensional magnitudes; indices (1) and (2) refer to the external zone and porous layer, respectively; \( \tilde{\mu}^{(1)} \) is the viscosity of the clear fluid, \( \tilde{\mu}^{(2)} \) denotes the effective viscosity of porous medium; \( k \) being the permeability of porous medium. The viscosity coefficients \( \tilde{\mu}^{(1)} \) and \( \tilde{\mu}^{(2)} \) are in general different.
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Here, \( \mathbf{v}^{(i)}, \mathbf{p}^{(i)}, i = 1, 2 \) be the velocity vector and pressure outside and inside the porous cylindrical shell respectively.

In dimensionless form; we put

\[
\begin{align*}
  r &= \frac{x}{b} , \quad u^{(1)} = \frac{q^{(1)}}{u} , \quad u^{(2)} = \frac{q^{(2)}}{u} , \quad p = \frac{\bar{p}}{b}, \quad \bar{p}_0 = \frac{\bar{u}}{b}, \quad l = \frac{\bar{a}}{b} , \quad \bar{c} = \frac{1}{\sqrt{\gamma}}, \quad \lambda = \frac{\mu_2}{\mu_1}, \quad \eta = \frac{1}{\sqrt{\kappa_0}}, \quad \alpha = \frac{\eta}{\lambda}, \quad z = \frac{z}{b} , \quad \bar{k} = k_0 \bar{r}^2
\end{align*}
\]

(4)

Here, \( \gamma \) is particle volume fraction; \( \lambda \) is Porosity.

From (2), Brinkman's equation in non dimensional form is

\[
\frac{d^2u^{(2)}}{dr^2} + \frac{1}{r} \frac{du^{(2)}}{dr} - \frac{a^2}{r^2} u^{(2)} = \frac{1}{\lambda^2} \frac{dp}{dz}
\]

(5)

Solution of equation (3) is

\[
\begin{align*}
  u^{(2)} &= c_1 r^\alpha + c_2 r^{-\alpha} + \frac{1}{\lambda^2(4-a^2)} \frac{dp}{dz} \bar{r}^2 ; \quad \alpha \neq 2
\end{align*}
\]

(6)

And

\[
\begin{align*}
  &u^{(2)} = c_1 r^2 + c_2 r^{-2} + \frac{1}{4\lambda^2} \frac{dp}{dz} r^3 ; \quad \alpha = 2
\end{align*}
\]

(7)

No slip boundary condition) implies \( u^{(2)} = 0 \) at \( r = 1 \)

So from eq. (4) & (5) we have;

\[
\begin{align*}
  &c_1 \lambda^2 + c_2 \lambda^{-2} + \frac{1}{\lambda^2(4-a^2)} \frac{dp}{dz} l^2 = 0 ; \quad \alpha \neq 2
\end{align*}
\]

(8)

\[
\begin{align*}
  c_1 \lambda^2 + c_2 \lambda^{-2} + \frac{1}{4\lambda^2} \frac{dp}{dz} l^3 ; \quad \alpha = 2
\end{align*}
\]

(9)

Again from eq. (1) and (4) gives the non dimensional form of Stokes' equation as;

\[
\begin{align*}
  \frac{1}{r} \frac{du^{(1)}}{dr} + \frac{a^2}{r^2} u^{(1)} = \frac{dp}{dz}
\end{align*}
\]

(10)

Its solution is given by;

\[
\begin{align*}
  u^{(1)} &= c_3 + c_4 \log r + \frac{1}{4\lambda^2} \frac{dp}{dz} \bar{r}^2
\end{align*}
\]

(11)

Continuity of velocity implies \( u^{(1)} = u^{(2)} \) at \( r = 1 \), eq. (4) and (8) gives

\[
\begin{align*}
  c_3 + \frac{1}{4a} \frac{dp}{dz} = c_1 + c_2 + \frac{1}{\lambda^2(4-a^2)} \frac{dp}{dz} \bar{r}^2 \quad \alpha \neq 2
\end{align*}
\]

(12)

At \( r = 1 \), continuity of stress i.e. \( \mu_1 \frac{du^{(1)}}{dr} = \mu_2 \frac{du^{(2)}}{dr} \) or \( \frac{du^{(1)}}{dr} = \lambda^2 \frac{du^{(2)}}{dr} \) implies

\[
\begin{align*}
  c_4 + \frac{\omega}{2} = \lambda^2 \left[ c_1 \alpha - c_2 \alpha + \frac{2\omega}{\lambda^2(4-a^2)} \right] \quad \text{for} \quad \alpha \neq 2
\end{align*}
\]

(13)

And

\[
\begin{align*}
  c_4 + \frac{\omega}{2} = \lambda^2 \left[ 2c_1 - 2c_2 + \frac{3\omega}{2\lambda^2} \right] \quad \text{for} \quad \alpha = 2
\end{align*}
\]

(14)

The dimensionless flow rate of fluid passing through the cylindrical cell is given by

\[
\begin{align*}
  Q &= 2\pi \int_1^1 u^{(2)} rdr + \int_1^1 u^{(1)} rdr
\end{align*}
\]

(15)

\[
\begin{align*}
  Q &= 2\pi \left[ \frac{c_1}{\alpha+2} (1 - l^{\alpha+2}) + \frac{c_2}{2-\alpha} (1 - l^{2-\alpha}) + \frac{\omega}{4\lambda^2(4-a^2)} (1 - l^4) \right] + 2\pi \left[ \frac{c_1}{\alpha} \left( \frac{1}{\gamma} - \frac{1}{\gamma^2} \right) + \frac{c_4}{4\gamma} \left( \log \frac{1}{\gamma} \right) - \frac{\omega}{4} \left( 1 - \frac{1}{\gamma} \right) \right]
\end{align*}
\]

(16)

In the considered case, the rate of filtration is equal to \( V_f = Q / (\pi / \gamma) \)
[6]. Using the Darcy law for the flow in porous medium, we obtain in dimensionless form

\[ V_f = -L_{11} dp / dz = -L_{11} \rho \]  

(17)

The dimensionless Hydrodynamic permeability is given by;

\[ L_{11} = -\frac{q_{r}}{\pi} \]

So we have

\[ L_{11} = -\frac{2r}{\omega} \left[ \frac{c_1}{\alpha+2} (1 - l^{\alpha+2}) + \frac{c_2}{2-\alpha} (1 - l^{2-\alpha}) + \frac{\omega}{4\lambda^2(4-\alpha^2)} (1 - l^4) \right] - \]

\[ \frac{2r}{\omega} \left[ -\frac{c_3}{2} \left( 1 - \frac{1}{\gamma} \right) + \frac{c_4}{4\gamma} \left( \log \frac{1}{\gamma} - \frac{\omega}{4} \left( 1 - \frac{1}{\gamma^2} \right) \right) \right] \]

For \( \alpha \neq 2 \)

\[ L_{11} = -\frac{2}{\omega} \left[ \frac{c_1}{4} (1 - l^4) - c_2 \log l + \frac{\omega}{20\lambda^2} (1 - l^5) \right] - \frac{2r}{\omega} \left[ -\frac{c_3}{2} \left( 1 - \frac{1}{\gamma} \right) + \frac{c_4}{4\gamma} \left( \log \frac{1}{\gamma} - \frac{\omega}{4} \left( 1 - \frac{1}{\gamma^2} \right) \right) \right] \]  

(18)

For \( \alpha = 2 \) using equation (7) as velocity and employing the same boundary conditions, we get

\[ L_{11} = -\frac{c_2}{\omega} \left( \log \frac{1}{\gamma} - \frac{\omega}{4} \left( 1 - \frac{1}{\gamma^2} \right) \right) \]

(19)

**Set of Solid Impermeable Particles**

At, \( \lambda \rightarrow \infty \), or \( l = 1 \) we have a particle without a porous layer that is absolutely impermeable to liquid; then, the expression for hydrodynamic permeability acquires the following form

\[ L_{11} = -\ln (\gamma) + \gamma^2 + \gamma - 2 \]

(20)

**Results and Discussions:** Figs.2 and 3 represents the influence of particle volume fraction on the hydrodynamic permeability of the membrane. Like the previous cases increase in \( \gamma \) causes decrease in \( L_{11} \) however, it is reported that \( L_{11} \) reports relatively less decay in comparison to homogeneous porous medium. Fig.3 shows the effect of viscosity ratio on \( L_{11} \). It has been reported that \( L_{11} \) attains lesser values comparative to homogeneous porous medium. Effect of viscosity ratio on hydrodynamic permeability of membrane is graphically discussed.

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**Fig. 1.** Physical model of the problem
Flow through a membrane of porous cylindrical particles

Fig. 2. Variation of dimensionless hydrodynamic permeability $L_{11}$ of a membrane consisting of cylindrical particles under longitudinal flow on particle volume fraction $\gamma$ for different cell models for $(\ell = 0.5, \alpha = 0.5, \lambda = 0.5)$.

Fig. 3. Variation of dimensionless hydrodynamic permeability $L_{11}$ of a membrane consisting of cylindrical particles under longitudinal flow on particle volume fraction $\gamma$ for different cell models for $(\ell = 0.5, \alpha = 1, \lambda = 0.5)$.

Fig. 4. Variation of dimensionless hydrodynamic permeability $L_{11}$ of a membrane consisting of cylindrical particles under longitudinal flow on viscosity ratio $\lambda$ for different cell models for $(\ell = 0.5, \eta = 1, \gamma = 0.5)$. 
References


Received: December, 2011