An Application of HAM for MHD Heat Source Problem with Variable Fluid Properties

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Abstract

The present study deals with the free convective flow of a viscous fluid past an isothermal vertical plate embedded in a homogeneous porous medium. It is assumed that both the viscosity and thermal conductivity vary linearly with temperature. The solution to the problem is obtained using Homotopy Analysis method (HAM) and the slip velocity values thus obtained are compared with the exact values obtained from the expression derived from the governing equations. It has been found that these values are in good agreement. Thus, the authors could present an approximate analytical solution to the class of MHD heat source problems which are more often solved using numerical methods.

Keywords: HAM, Heat source, Magnetic field, Variable fluid properties

I. Introduction

Heat transfer problems in porous media, due to their enormous engineering applications, have attracted researchers from diversified fields for the past few decades. Attempts have been made to understand the effect of fluid properties, properties of the porous media on the flow and the heat transfer. Classical books on heat transfer [4,14] provide a great insight into this, where solutions (analytical or mostly numerical) to the problems have been found under some valid assumptions such as: the fluid viscosity is a constant, thermal conductivity of medium is a constant and so on. This is because the governing heat flow equations are highly nonlinear,

that, in general cannot be solved analytically or even numerically. Later, attempts were made to find more realistic solutions by assuming variable fluid properties etc. However, most of the solutions were sought out using numerical methods as can be seen in for instance in references [5,9]. Further, few other engineering applications require the study of the effect of magnetic field and the presence of heat source in the flow domain. Here again, researchers found mostly numerical solutions only [3, 12, 10, 15, 16]. But, with the advent of approximate analytical methods like Adomian decomposition method, Homotopy perturbation method, Homotopy analysis method, it is now possible to find analytical solutions to nonlinear equations as well [17].

Thus, the present study considers the heat transfer problem with variable viscosity and thermal conductivity under the presence of magnetic field and heat source. Though, in general, all the properties of the fluids change with temperature, in fact, changes in viscosity and thermal conductivity are quite significant for many fluids and hence their effect on the heat flow and fluid flow are studied in the present paper. An approximate analytical solution using the Homotopy analysis method (HAM) introduced by Liao in 1992 [18, 2] is found for the problem considered. Although several other approximate analytical methods (listed above) are available, the authors chose to work with this method as it has been found to be a very successful method in finding solutions to highly nonlinear equations and coupled equations [2,1,6]. References [1, 7, 13, 8] emphasize the effectiveness of this method in solving a few heat transfer problems as well.

II. Mathematical formulation of the problem

Consider an isothermal vertical flat plate embedded in a saturated porous medium with viscous incompressible fluid. The porous medium is assumed to be homogeneous and is in the thermal equilibrium with the surrounding fluid in the presence of heat source Q and a magnetic field of uniform strength is assumed to be acting in a direction normal to the plate and transverse to the vertical plate.

Let x -axis be taken vertically along the plate and y -axis perpendicular to it, T_w is assumed as temperature of the plate and T_∞ is an ambient temperature of the fluid. The orientation of the plate for both hot and cold plate for free convection is presented in the Fig-1.

The equations governing the free convection boundary-layer flow are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$\frac{\partial p}{\partial x} + (\rho - \rho_\infty)g + \sigma B_0^2 u + \frac{\mu}{K} u = 0 \quad (2)$$

$$\frac{\partial p}{\partial y} + \frac{\mu}{K} v = 0 \quad (3)$$

$$\rho c_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = \frac{\partial}{\partial y} \left(k_m \frac{\partial T}{\partial y} \right) + Q_0 (T - T_\infty) \quad (4)$$

where u, v are fluid velocity components, T is fluid temperature, K is Permeability, k_m is effective thermal conductivity of the porous medium, B_0 is the magnetic flux.

Boussinesq approximation $\rho = \rho_\infty [1 - \beta(T - T_\infty)]$ is taken as the body force term and Q is the heat source.

Now expressing the velocity components in terms of the stream function, we have: $u = \frac{\partial \psi}{\partial y}$

$$\text{and } v = -\frac{\partial \psi}{\partial x}.$$

After introducing the stream function ψ and eliminating fluid pressure from equations (2) and (3), the governing equations simplify to

$$(\mu + K \sigma B_0^2) \left(\frac{\partial^2 \psi}{\partial y^2} \right) + \left(\frac{\partial \psi}{\partial y} \right) \left(\frac{\partial \mu}{\partial y} \right) = K g \rho_\infty \beta \left(\frac{\partial T}{\partial y} \right) \quad (5)$$

$$\frac{\partial \psi}{\partial y} \frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial T}{\partial y} = \frac{1}{\rho c_p} \left(k_m \frac{\partial^2 T}{\partial y^2} + \frac{\partial k_m}{\partial y} \frac{\partial T}{\partial y} \right) + \frac{Q_0 (T - T_\infty)}{\rho c_p} \quad (6)$$

The boundary conditions on T and ψ are

$$\left. \begin{array}{l} \text{at } y = 0, T = T_w, \frac{\partial \psi}{\partial x} = 0, \\ \text{as } y \rightarrow \infty, T \rightarrow T_\infty, \frac{\partial \psi}{\partial y} \rightarrow 0 \end{array} \right\} \quad (7)$$

Introducing **Rayleigh** number (Ra_x), **Hartman** number (M^2), a magnetic interaction parameter C , a heat source parameter Q and the non-dimensional functions f , θ together with a similarity variable η through the relations

$$\left. \begin{array}{l} Ra_x = \frac{\rho_\infty K g \beta |T_w - T_\infty| x}{\alpha_m \mu_f}; M^2 = \frac{B_0^2 L^2 \sigma}{\mu_f}; K^* = \frac{L^2}{K}; C = \frac{K^*}{K^* + M^2}; \\ Q = \frac{Q_0 x^2}{k_m (T_w - T_\infty) Ra_x}; f(\eta) = \frac{\psi}{\alpha Ra_x^{\frac{1}{2}}}; \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}; \eta = \frac{y}{x} Ra_x^{\frac{1}{2}} \end{array} \right\} \quad (8)$$

Equations (5), (6) are rewritten as

$$\left[1 + C\gamma_\mu\left(\theta - \frac{1}{2}\right)\right]f'' + C\gamma_\mu f'\theta' = C\theta' \quad (9)$$

$$\left[1 + \gamma_k\left(\theta - \frac{1}{2}\right)\right]\theta'' + \frac{1}{2}f\theta' + \gamma_k\theta'^2 + Q\theta = 0 \quad (10)$$

where γ_μ is the viscosity variation coefficient and γ_k is the thermal conductivity variation coefficient.

Taking $\mu = \mu_f \left[1 + \gamma_\mu\left(\theta - \frac{1}{2}\right)\right]$ and $k_m = k_f \left[1 + \gamma_k\left(\theta - \frac{1}{2}\right)\right]$ (Ref [5],[10][15]).

The boundary condition (7) becomes

$$\left. \begin{array}{l} \text{at } \eta = 0, \quad \theta = 1, \quad f = 0, \\ \text{as } \eta \rightarrow \infty, \quad \theta \rightarrow 0, \quad f' \rightarrow 0 \end{array} \right\} \quad (11)$$

Equation (9) can be integrated once using the condition on f' at infinity to get

$$f' = \frac{C\theta}{\left[C\gamma_\mu\left(\theta - \frac{1}{2}\right) + 1\right]} \quad (12)$$

Evaluating this expression at $\eta = 0$ which gives the analytical expression for the slip velocity $f'(0)$ as

$$f'(0) = \frac{2C}{C\gamma_\mu + 2} \quad (13)$$

III. Parameters of the Problem

In the present problem, the flow and heat transfer depend on the parameters: C (the magnetic field parameter), γ_μ (the viscosity variation coefficient), γ_k (thermal conductivity variation coefficient) and Q (the heat source parameter). The magnetic parameter C , takes on smaller values (less than unity) when either the porous parameter takes smaller values or the Hartmann number takes larger values. It is the case when porosity of the medium or the intensity of magnetic field is high. Further, when there is no applied magnetic field, M^2 takes a zero value as a result of which C takes the value ‘unity’ as can be seen from equation (8). So, in the present study, solutions are found for the values of $C = 0.1, 0.5$ and 1 .

As in [10] and [15], solutions has been found for the two cases

(i) $T_w > T_\infty$ (Hot Plate) for which, as the temperature increases, μ increases ($\mu_w < \mu_\infty$) whereas k_m decreases ($k_w > k_\infty$) as a result of which $\gamma_\mu < 0, \gamma_k > 0$

(ii) $T_w < T_\infty$ (Cold Plate) for which as the temperature increases, μ decreases ($\mu_w > \mu_\infty$) while k_m increases ($k_w < k_\infty$) and hence we have that $\gamma_\mu > 0$, $\gamma_k < 0$.

In this paper solutions are found for the values of both γ_μ and γ_k as ± 0.2 . The heat source parameter Q takes on positive values and the present study takes up the values as $Q = 0.1$ and 0.2 .

IV. HAM

Consider a nonlinear differential equation of the form:

$$N(u(x)) = 0 \quad (14)$$

where N is a nonlinear operator, x is the independent variable and $u(x)$ is the unknown function. Let $u_0(x)$ be the initial approximation of the exact solution $u(x)$ and L be an auxiliary linear operator with the property that

$$L(f) = 0 \text{ when } f = 0. \quad (15)$$

In this method, we construct the Homotopy which is a continuous mapping $H : u(x) \rightarrow \phi(x; q)$ defined as

$$H(\phi(x; q); q) = (1 - q)L(\phi(x; q) - u_0(x)) - hH(x)qN(\phi(x; q)) \quad (16)$$

Here $H(x)$ is an auxiliary function and h is an auxiliary parameter called the convergence control parameter, $q \in [0, 1]$ is an embedding parameter and $\phi(x; q)$ is the approximate solution to the given problem. We notice from equation (16) that the solution obtained using this method, depends on the four important factors namely the initial approximation $u_0(x)$, the linear operator L , the auxiliary function $H(x)$ and the auxiliary parameter h .

Taking $q = 0$ and the Homotopy defined by equation (16) to be zero, the so called zeroth order deformation equation is given by

$$L(\phi(x; 0) - u_0(x)) = 0 \quad (17)$$

In view of the linearity of the operator ' L ', the zeroth deformation equation is given by

$$\phi(x; 0) = u_0(x) \quad (18)$$

Now, when $q = 1$, equation (16) takes the form

$$N(\phi(x; 1)) = 0$$

This equation is same as the given equation provided

$$\phi(x;1) = u(x) \quad (19)$$

This shows that as the embedded parameter q varies from 0 to 1, $\phi(x;q)$ varies from the initial guess $u_0(x)$ (as is seen in equation (18)) to the exact solution $u(x)$ (as seen equation (19)).

Define the m^{th} order deformation derivatives as

$$u_0^{[m]}(x) = \frac{\partial^m}{\partial q^m}(\phi(x;q)) \Big|_{q=0} \quad (20)$$

Then, using the Taylor's theorem, $\phi(x;q)$ can be expanded as a power series of q as

$$\phi(x;q) = \phi(x;0) + \sum_{m=1}^{\infty} \frac{u_0^{[m]}(x)}{m!} q^m \quad (21)$$

Writing $u_m(x) = \frac{u_0^{[m]}(x)}{m!}$ and using equation (18), the above takes the form

$$\phi(x;q) = u_0(x) + \sum_{m=1}^{\infty} u_m(x) q^m \quad (22)$$

With suitable choice of the initial guess, the auxiliary linear operator, the convergence control parameter and the auxiliary function, Liao proved that the above power series solution converges for $q = 1$ [17].

Now, to find the solution using equation (22), the functions $u_m(x)$ are to be found for $m=1,2,3,\dots$

Liao [17] has derived that these functions are given by the m^{th} order deformation equation defined as follows:

$$L(u_m(x) - \chi_m u_{m-1}(x)) = hH(x) R_m(\tilde{u}_{m-1}(x)) \quad (23)$$

where

$$R_m(\tilde{u}_{m-1}(x)) = \frac{1}{(m-1)!} \left(\frac{\partial^{m-1}}{\partial q^{m-1}} (N(\phi(x;q))) \right) \Big|_{q=0} \quad (24)$$

$$\text{and } \chi_m = \begin{cases} 0, m \leq 1 \\ 1, \text{otherwise} \end{cases} \quad (25)$$

After determining $u_m(x)$ for $m=1,2,\dots$, an approximate solution to the problem given in (14) is

$$u(x) \approx u_0(x) + \sum_{m=1}^{\infty} u_m(x) \quad (26)$$

Solution of equations (9) and (10):

$$\begin{aligned} \text{Let } L_f &\equiv f'' - C\theta' \\ L_\theta &\equiv \theta'' + \theta' \end{aligned} \quad (27)$$

$$\begin{aligned} f_0(\eta) &= C(1 - e^{-\eta}) \\ \theta_0(\eta) &= e^{-\eta} \end{aligned} \quad (28)$$

$$\begin{aligned} \text{Here } N_f &\equiv \left[1 + C\gamma_\mu \left(\theta - \frac{1}{2}\right)\right] f'' + C\gamma_\mu f' \theta' - C\theta' = 0 \\ N_\theta &\equiv \left[1 + \gamma_k \left(\theta - \frac{1}{2}\right)\right] \theta'' + \frac{1}{2} f \theta' + \gamma_k \theta'^2 + Q\theta = 0 \end{aligned} \quad (29)$$

Construct the homotopy for the above nonlinear coupled equations as:

$$\begin{aligned} H(f(\eta; q); q) &= (1-q)L(f(\eta; q) - f_0(\eta)) - hN_f(f(\eta; q)) \\ H(\theta(\eta; q); q) &= (1-q)L(\theta(\eta; q) - \theta_0(\eta)) - hN_\theta(\theta(\eta; q)) \end{aligned} \quad (30)$$

Assume the solutions as

$$\begin{aligned} f(\eta; q) &= f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta) q^m \\ \theta(\eta; q) &= \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta) q^m \end{aligned} \quad (31)$$

where $f_m(\eta)$ and $\theta_m(\eta)$ are obtained using equations (23)- (25).

Using MATHEMATICA, the first four approximations for $f(\eta)$ and $\theta(\eta)$ are calculated and the plots are presented.

Results and Discussions**COLD PLATE: For $\gamma_\mu > 0$ and $\gamma_k < 0$**

To find the values of the convergence control parameter ‘ h ’ in equations (30), the h -graphs for f i.e $f'(h)$ and $f''(h)$ are plotted at $\eta = 0$ in Fig 2. It can be seen that for $C=0.1$ and $Q=0.1$ the control parameter $h=-0.75$.

Similarly, to find the convergence control parameter h for finding θ , the h -graphs for θ are plotted as in Fig 3.

The values of the slip velocity $f'(0)$ are found for a particular set of values of ‘ C ’ and ‘ Q ’ using the HAM solutions and also using the equation (12). The values are found to be in good agreement with each other.

A point to be noted here: Equation (12) shows that the slip velocity i.e., the velocity at the vicinity of the plate is independent of ' Q ' and the same can be observed from tables (1) and (2). Though HAM uses the value of ' Q ' in the evaluation of the $R_m(\vec{u}_{m-1}(x))$ as shown in equations (24) and (29), but still the values continued to be a constant for different values of ' Q '. This emphasizes the fact that HAM is an efficient tool for solving coupled nonlinear equations.

HOT PLATE: For $\gamma_\mu < 0$ and $\gamma_k > 0$:

As discussed above for the hot plate, the h curves are drawn for calculating the convergence control parameter ' h ' for f and θ , the slip velocity values calculated using equation (12) and the HAM solution are tabulated in table 2.

The values of slip velocities with distinct parameters are shown in the table 1 and table 2 for cold as well as hot plate.

- (i) It is apparent that the magnetic field lines obstruct the flow of fluid and hence reduced flow can be expected with the increase in the intensity of the magnetic field. The same has been observed here: slip velocity decreased with the increase in the intensity of the magnetic field (i.e., C changes from 1 to 0.1).
- (ii) It can also be seen that the values of the slip velocity will not vary with the strength of the heat source.
- (iii) Further, tables (1) and (2) show that the value of slip velocity $f'(0)$ is larger for hot plate (i.e., $\gamma_\mu = -0.2$, $\gamma_k = 0.2$) than cold plate (i.e., $\gamma_\mu = 0.2$, $\gamma_k = -0.2$).
- (iv) The temperature profiles are presented in fig 5 and fig 9, it can be observed that thermal boundary layer thickness for cold plate is more than hot plate.

Conclusion

In the present study, the approximate analytical solution using HAM is found for the hot as well as the cold plate in presence of heat source and magnetic field. The slip velocity values calculated using the approximate method are compared with the exact values and are found to be in good agreement with each other. The slip velocity for hot plate is more than cold plate and the thermal boundary layer thickness for the cold plate is more than the hot plate.

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Figures

Fig 6 Variations of $f'(h)$ and $f''(h)$ for $Q=0.1, C=0.1, \gamma_k < 0$ and $\gamma_\mu > 0$ ($-1.5 < h < 1$)

Fig7 Variations of $\theta'(h)$ and $\theta''(h)$ for $Q=0.1, C=0.5, \gamma_k < 0$ and $\gamma_\mu > 0$ ($-1.5 < h < 0.5$)

Fig 8 Variations of $f(\eta)$ for $Q=0.1, C=0.5, \gamma_k < 0$ and $\gamma_\mu > 0$

Fig 9 Variations of $\theta(\eta)$ for $Q=0.1, C=0.5, \gamma_k < 0$ and $\gamma_\mu > 0$

Table 1 Values of Slip velocity for $\gamma_\mu = 0.2$ and $\gamma_k = -0.2$

C	Solutions	Q=0.1	Q=0.2
0.1	Exact Solution	0.09901	0.09901
	HAM Solution	0.09849999999999992	0.09901483416450818
0.5	Exact Solution	0.476191	0.476191
	HAM Solution	0.47468450260414824	0.47670798020498795

Table 2 Values of Slip velocity for $\gamma_\mu = -0.2$ and $\gamma_k = 0.2$

C	Solutions	Q=0.1	Q=0.2
0.1	Exact Solution	0.1010101	0.1010101
	HAM Solution	0.10113030143333651	0.10113324373334337
0.5	Exact Solution	0.5263158	0.5263158
	HAM Solution	0.5262226250000012	0.5262226250000012